KEY CONCEPTS

1. Basic Theorems & Results of Triangles

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both condition are independent & necessary) In case of a triangle, any one of the condition is sufficient, other satisfies automatically.

(b) **Thales Theorem (Basic Proportionality Theorem):** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.

Converse : If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) **Similarity Theorem :**

(i) **AAA similarity :** If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) **SSS similarity :** If the corresponding sides of two triangles are proportional, then they are similar.

(iii) **SAS similarity :** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular

(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

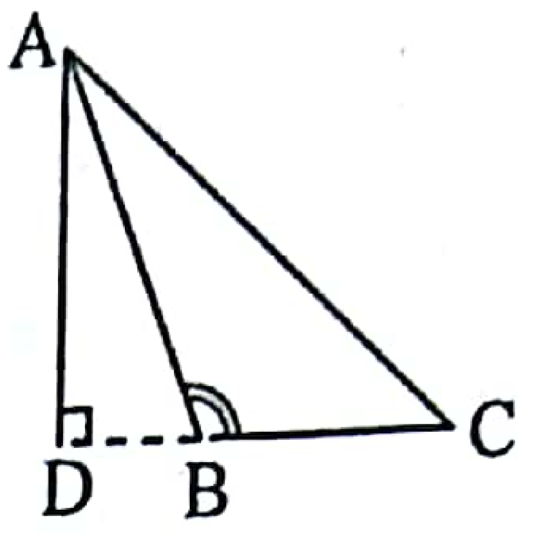
(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

(d) **Pythagoras Theorem :**

(i) In a right triangle the square of hypotenuse is equal to the sum of square of the other two sides.

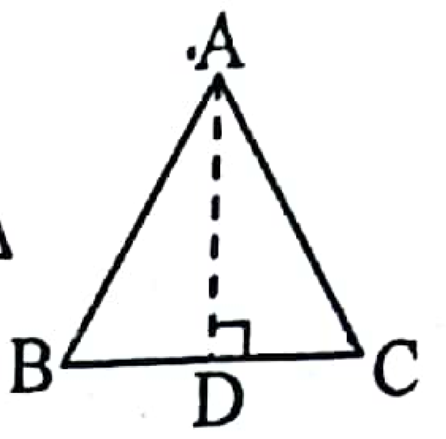
Converse : In a triangle if square of one side is equal to sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

(ii) In obtuse Δ



$$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$$

(iii) In Acute Δ



$$AC^2 = AB^2 + BC^2 - 2BC \cdot BD$$

(e) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true)

(f) The line joining the mid points of two sides of a Δ is parallel & half of the third side. (It's converse also true)

(g) (i) The diagonals of a trapezium divided each other proportionally. (converse is also true)

(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally.

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on transversals are proportional.

(h) In any triangle the sum of squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

(i) In any triangle the three times the sum of squares of the sides of a triangle is equal to four times the sum of the square of the medians of the triangle.

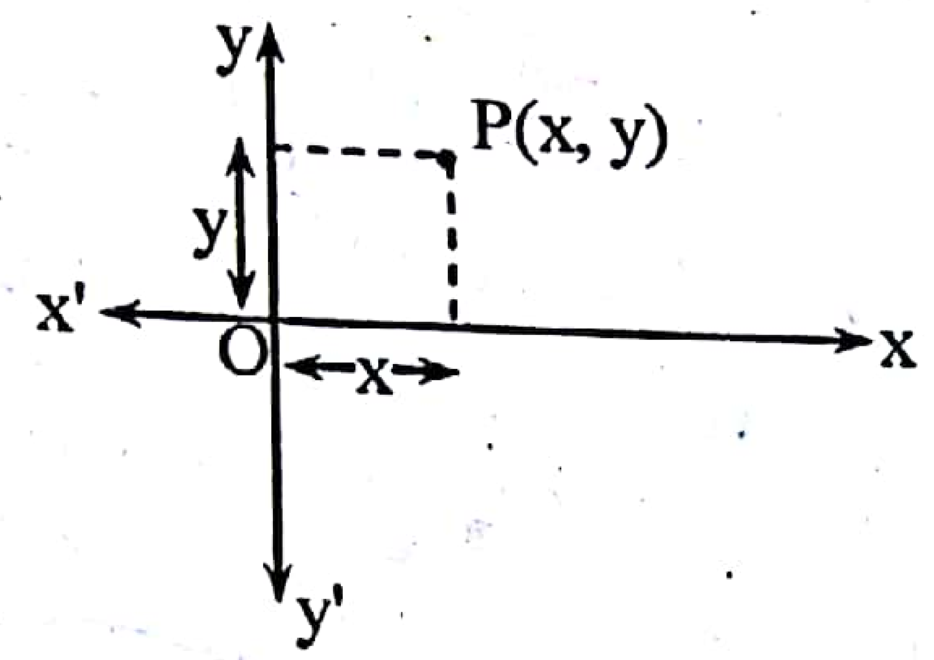
(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

2. Introduction of coordinate geometry

Coordinate geometry is a combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician Rene Descartes. The resulting combination of analysis and geometry is referred as **analytical geometry.**

3. Cartesian co-ordinates

In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y-axis above the origin are positive : distances below are negative.

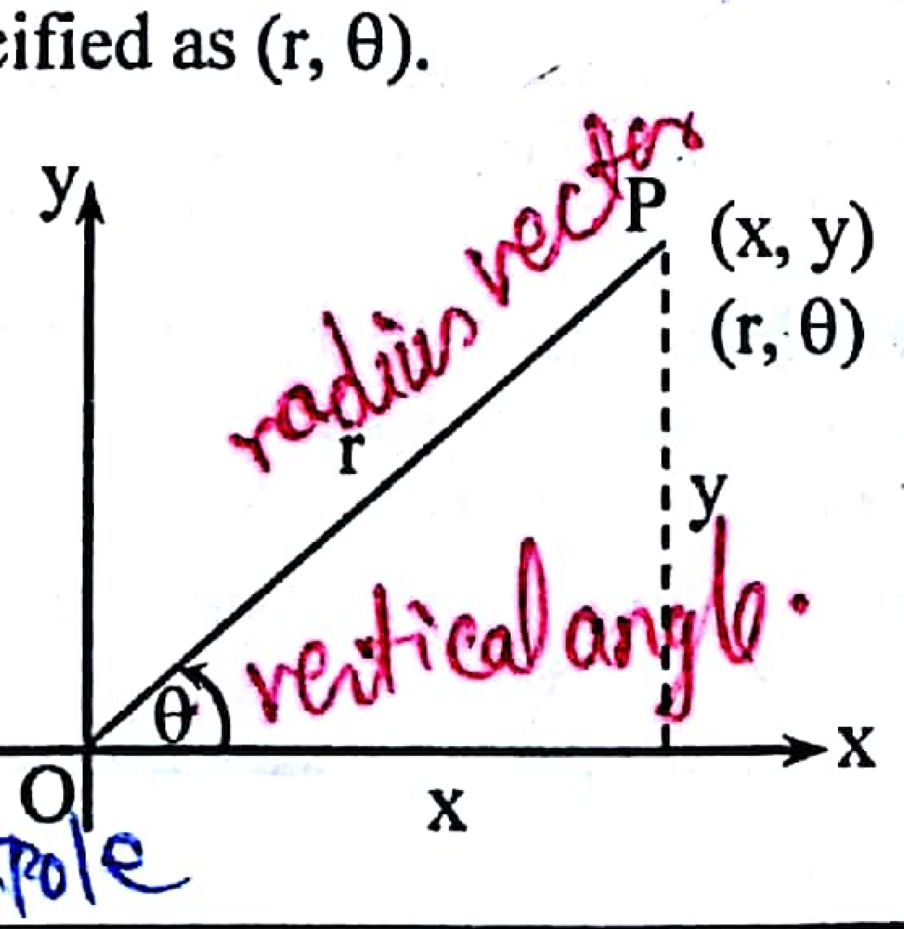


The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). They y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

4. Polar co-ordinates

A coordinate system in which the position of a point is determined by the length of a line segment from fixed origin together with the angle or angles that the line segment makes with a fixed line or lines. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).



If (x, y) are Cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

5. Distance formula and its applications

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note :

(a) Three given points A, B and C are collinear, when sum of any two distances out of AB, BC, CA is equal to the remaining third otherwise the points will be the vertices of triangle.

(b) Let A, B, C & D be the four given points in a plane. Then the quadrilateral will be :

(i) Square if AB = BC = CD = DA & $AC = BD$; $AC \perp BD$

(ii) Rhombus if AB = BC = CD = DA & $AC \neq BD$; $AC \perp BD$

(iii) Parallelogram if AB = DC, BC = AD; $AC \neq BD$; $AC \not\perp BD$

(iv) Rectangle if AB = CD, BC = DA, $AC = BD$; $AC \not\perp BD$

Quadrilateral	Diagonals	Angle between diagonals
(i) Parallelogram	Not equal	$\theta \neq \frac{\pi}{2}$
(ii) Rectangle	Equal	$\theta \neq \frac{\pi}{2}$
(iii) Rhombus	Not equal	$\theta = \frac{\pi}{2}$
(iv) Square	Equal	$\theta = \frac{\pi}{2}$

Note :

(i) Diagonal of square, rhombus, rectangle and parallelogram always bisect each other.

(ii) Diagonal of rhombus and square bisect each other at right angle.

5.1 Position of three points :

Let A, B, C are points lie in a plane then two condition arises.

5.1.1 Collinearity of three given points :

The three given points A, B, C are collinear i.e., lie on the same straight line, if

(i) any of the three points (say B) lie on the straight line joining the other two points.

Note :

The other conditions for collinearity is

(i) area of ABC is zero

$$\text{It means } \Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)] = 0$

(ii) Slope of AB = slope of BC

Slope Formula (H)

If θ is the anticlockwise angle at which a straight line is inclined to the positive direction of the x-axis, and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$.

Note :

(i) If θ is 90°, m does not exist, but the line is parallel to the y-axis.

(ii) If θ = 0, then m = 0 and the line is parallel to the x-axis.

(iii) If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by: $m = (y_1 - y_2)/(x_1 - x_2)$

* We see later these formula

6. Section formula

Coordinates of a point which divides the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ are

(i) $$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

(internal division)

(ii) $$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

(external division)

When m_1, m_2 are of opposite sign, the division is external.

Note :

(i) Co-ordinates of any point on the line segment which divide the line joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio of λ : 1 is given by

$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}\right), (\lambda \ne -1) \text{ (Imp.)}$$

(ii) Lines formed by joining (x_1, y_1) and (x_2, y_2) is divided by

(a) X-axis in the ratio $\Leftrightarrow -y_1/y_2$

(b) Y-axis in the ratio $\Leftrightarrow -x_1/x_2$

If ratio is positive the axis divide it internally and if negative divides externally.

(iii) Line Ax + By + C = 0 divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio λ : 1, then

$$\lambda = -\left(\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C}\right)$$

If λ is positive it divides internally if λ is negative then externally.

(iv) If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically, $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$

i.e. AP, AB & AQ are in H.P.

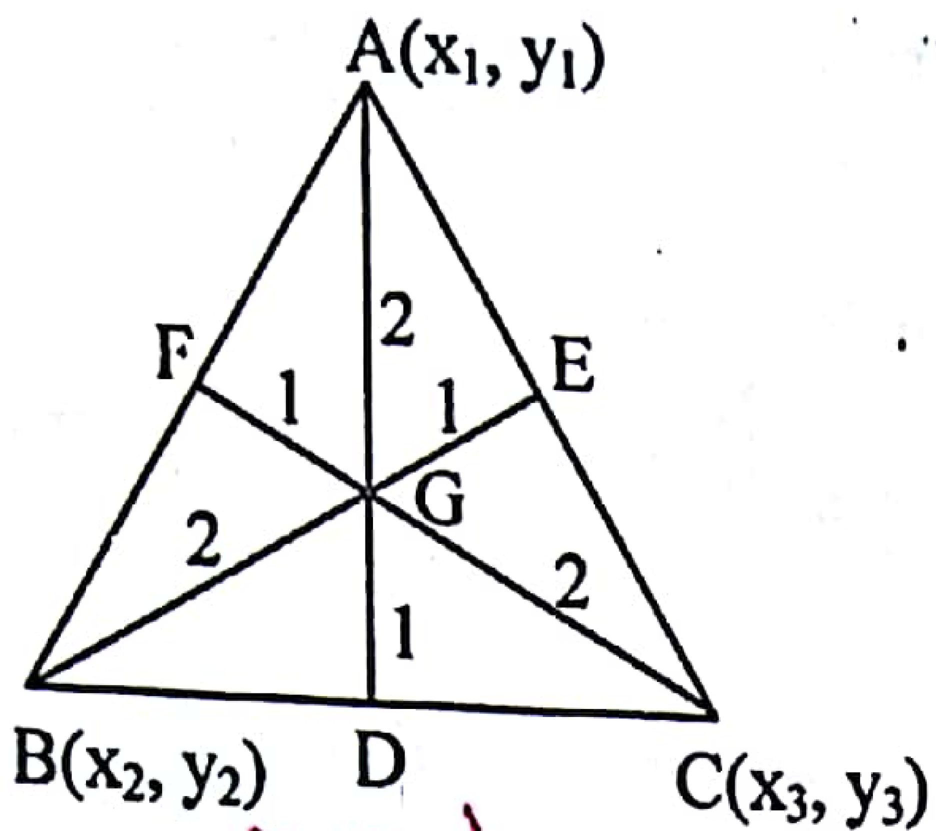
7. Co-ordinates of some particular points

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

7.1 Centroid : (The point of Intersection of median of a triangle) (G)

The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices) centroid divides the median in the ratio of 2 : 1. Coordinates of

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

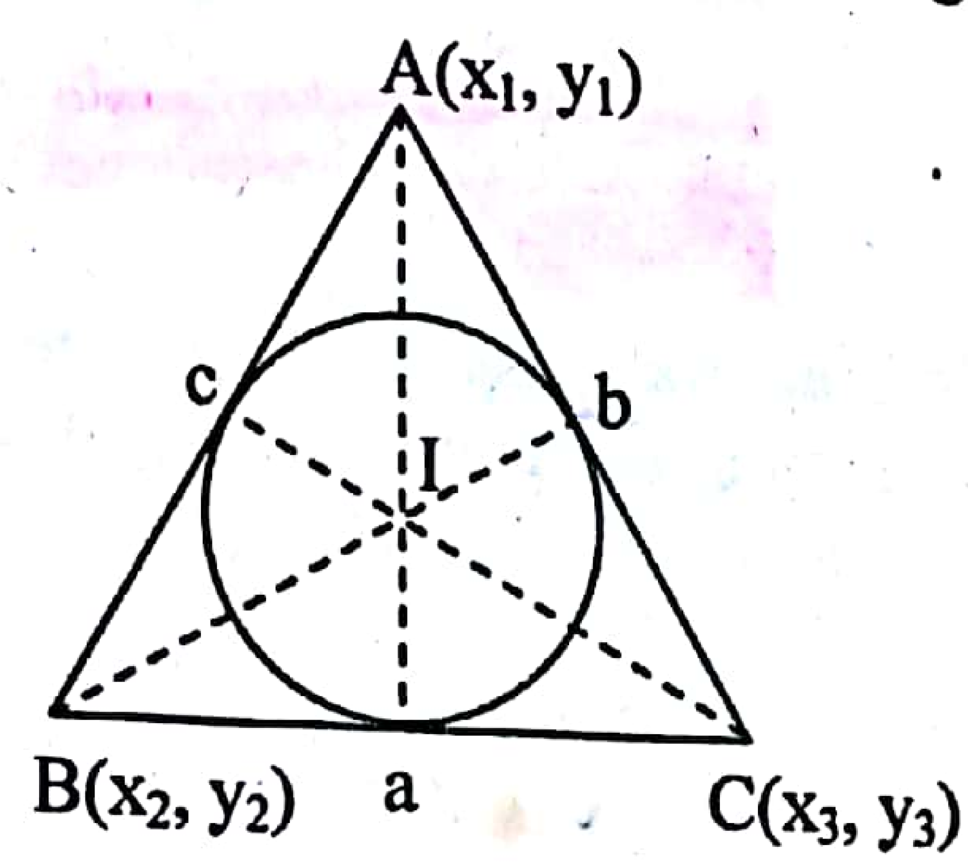


7.2 Incentre :

The incentre of the triangle is the point of intersection of internal bisector of the angle. Also it is a centre of a circle touching all the sides of a triangle. Co-ordinates of incentre

$$I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

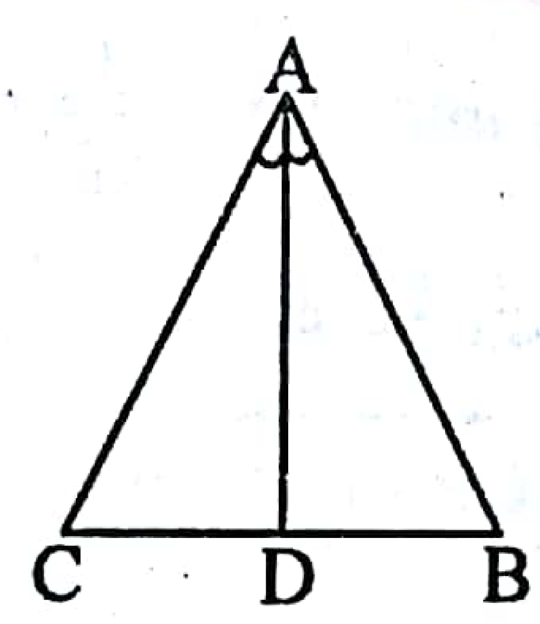
where a, b, c are the sides of triangle ABC



Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides

eg. $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$



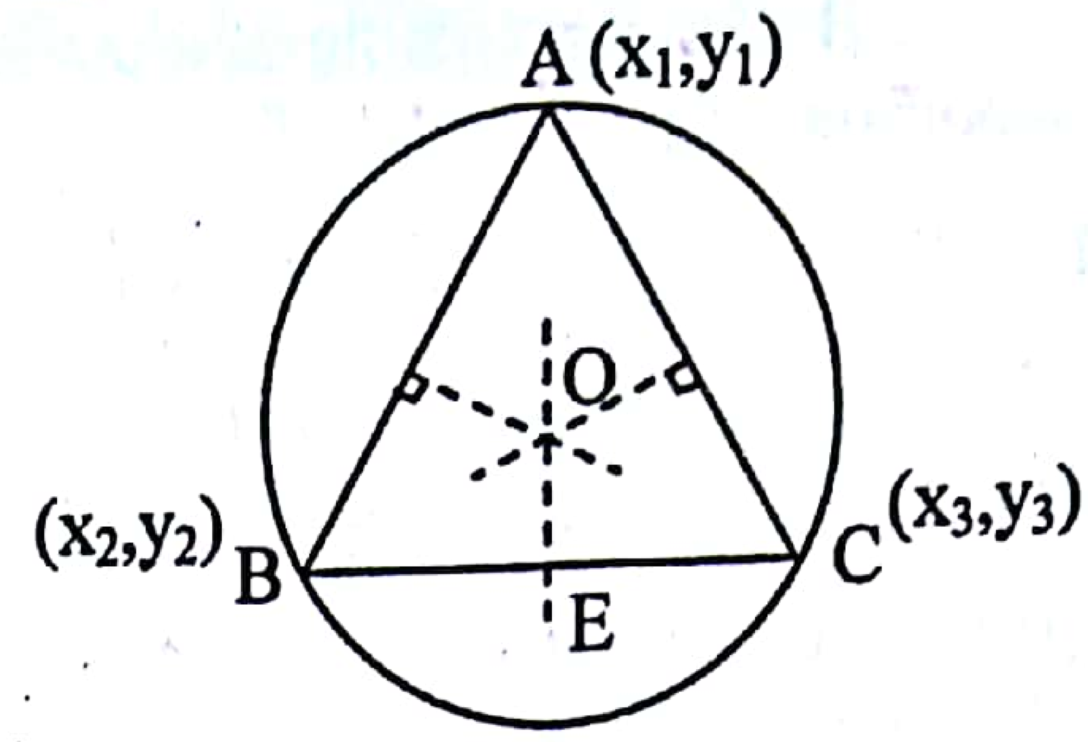
(ii) Incentre divides the angle bisector in the ratio

(b + c) : a, (c + a) : b and (a + b) : c

7.3 Circumcentre :

It is the point of intersection of perpendicular bisectors of the sides of the triangle. It is also the centre of a circle passing through the vertices of the triangle. If O is the circumcentre of any ΔABC.

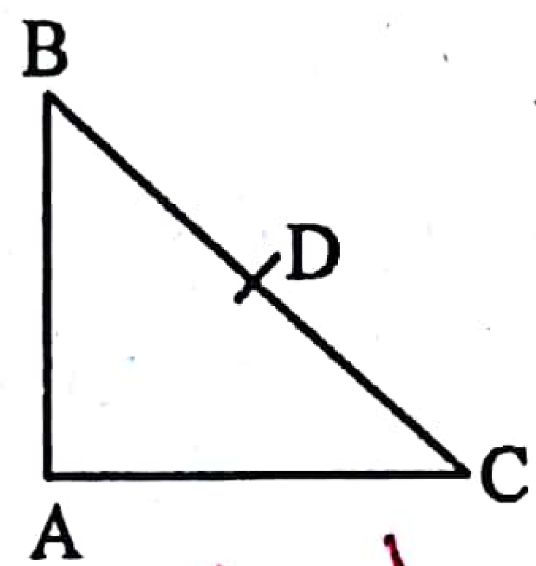
Then $OA^2 = OB^2 = OC^2$



Note:

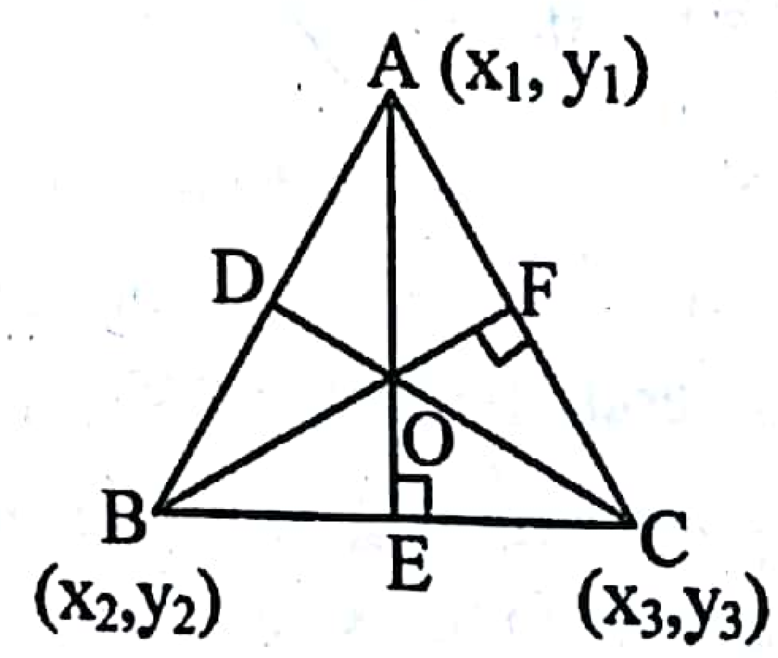
If a triangle is right angle, then its circumcentre is the mid point of hypotenuse.

For Example : in a right angle triangle ABC the point D is known as the circumcentre of a triangle



7.4 Orthocentre :

It is the point of intersection of perpendicular drawn from vertices on opposite sides called altitudes of a triangle and can be obtained by solving the equation of any two altitudes.



about orthocentre of Δ we will be discuss later.

Note :

If a triangle is right angle triangle, then ortho centre is the point where right angle is formed.

Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre coincides.

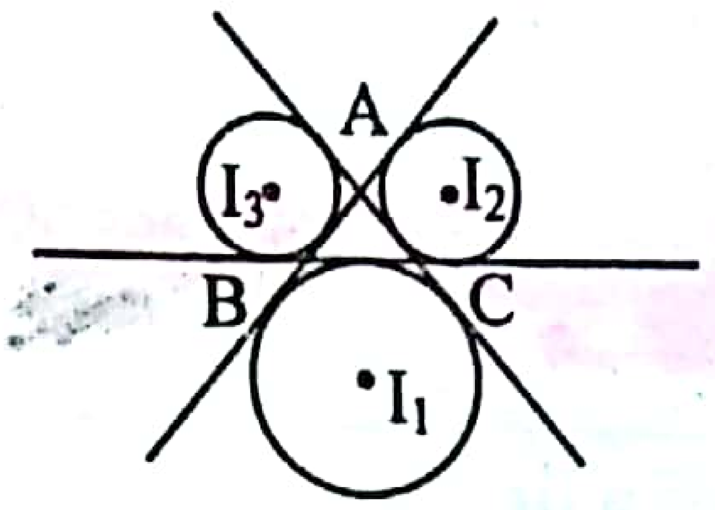
(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1.

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

7.5 Ex-centres :

The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$



Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively, and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

8. Area of triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix}$$

$$= \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle**

If altitude of any equilateral triangle is p, then its area = $\frac{p^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area = $\left(\frac{a^2\sqrt{3}}{4}\right)$

9. Locus

It is the path or the curve traced by a moving point satisfying the given condition.

9.1 Equation to the locus of a point :

The equation to the locus of a point is the algebraic relation which is satisfied by the coordinates of every point on the locus of the point.

Step taken to find the equation of locus

I. Assumes the coordinate of the point say (h, k) whose locus is to be find.

II. Write the given condition involving (h, k)

III. Eliminate the variable(s) if any

IV. Replace $h \longrightarrow x$ and $k \longrightarrow y$

The equation so obtained is the locus of the point which moves under some definite condition.

10. Miscellaneous points

(a) Reflection (Image) of a point :

Let (x, y) be any point then its image with respect to

(i) x axis $\Rightarrow$ $(x, -y)$

(ii) y axis $\Rightarrow$ $(-x, y)$

(iii) origin $\Rightarrow$ $(-x, -y)$

(iv) Line $y = x$ $\Rightarrow$ (y, x)

(b) A triangle is isosceles if any two of its median are equal.

(c) Triangle having integral coordinate can never be equilateral.

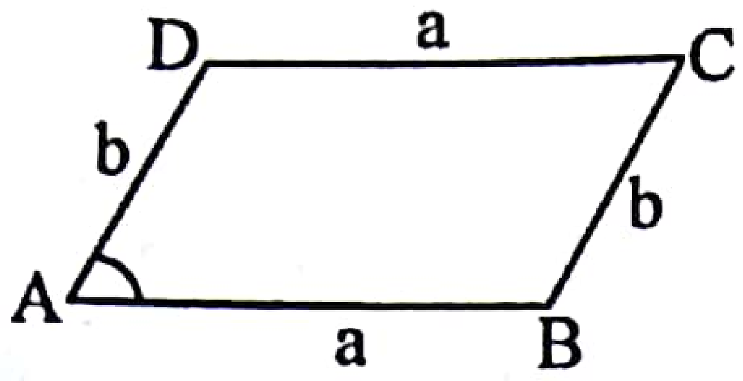
(d) If $a_rx + b_ry + c_r = 0$ ($r = 1, 2, 3$) are the sides of a triangle then the area of the triangle is given by

$$\frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

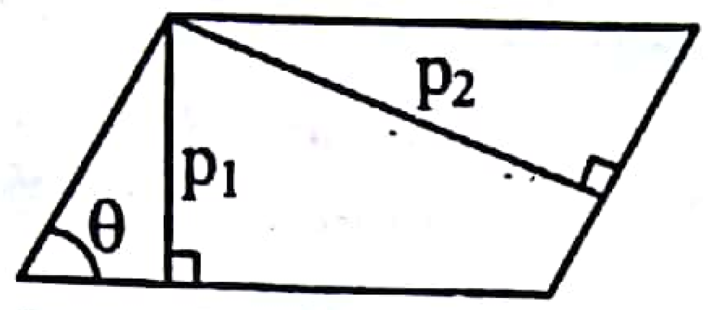
where C_1, C_2, C_3 are the cofactor of c_1, c_2, c_3 in the determinant.

(e) **Area of parallelogram**

(i) Whose sides are a and b and angle between them is θ is given by = ab sinθ



(ii) Whose length of perpendicular from one vertices to the opposite sides are p_1 and p_2 and angle between sides is θ is given by $= \dfrac{p_1 p_2}{\sin\theta}$



(f) A triangle having vertices $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ then

area of $\Delta = a^2[(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)]$

(g) Area of Δ formed by coordinate axis and the line $ax + by + c$ is $\dfrac{c^2}{2ab}$

(h) Area of rhombus formed by

$|ax| + |by| + c = 0$ is $\dfrac{2c^2}{ab}$

(i) Three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if slope of AB = slope of BC.

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(j) To remove the term of xy in the equation $ax^2 + 2hxy + by^2 = 0$. The angle θ through which the axis must be turned

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2h}{a-b}\right).$$

11. Definition of a straight line

A straight line is a curve such that every point on the line segment joining any two points on it lie on it.

12. Equation of straight line

A relation between x & y which is satisfied by coordinates of every point lying on a line is called the equation of straight line. Every first degree equation in x, y i.e. $ax + by + c = 0$ represents a line.

Thus a line is also defined as the locus of a point satisfying the condition $ax + by + c = 0$ where a, b, c are constant.

13. Equation of straight line parallel to axes

(i) Equation of x-axis $\Rightarrow y = 0$

(ii) Equation of a line parallel to x-axes at a distance of b $\Rightarrow y = b$

(iii) Equation y-axis $\Rightarrow x = 0$

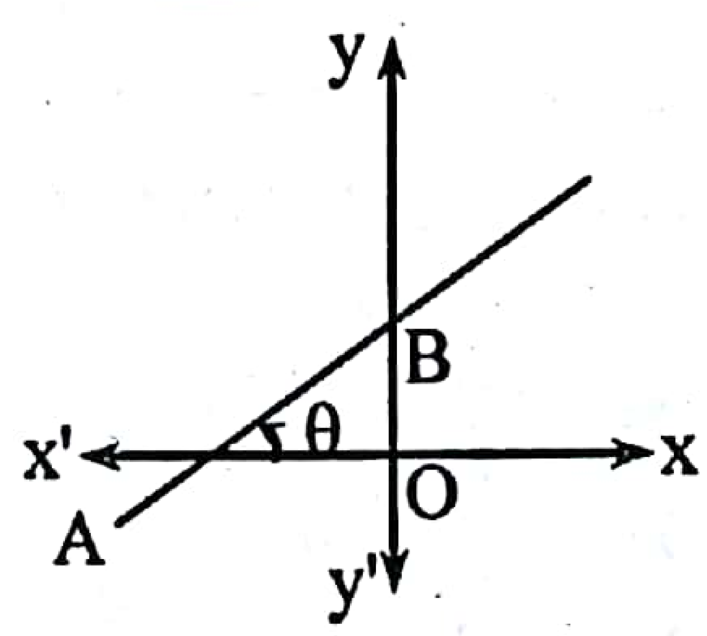
(iv) Equation of a line parallel to y-axis and at a distance of a $\Rightarrow x = a$

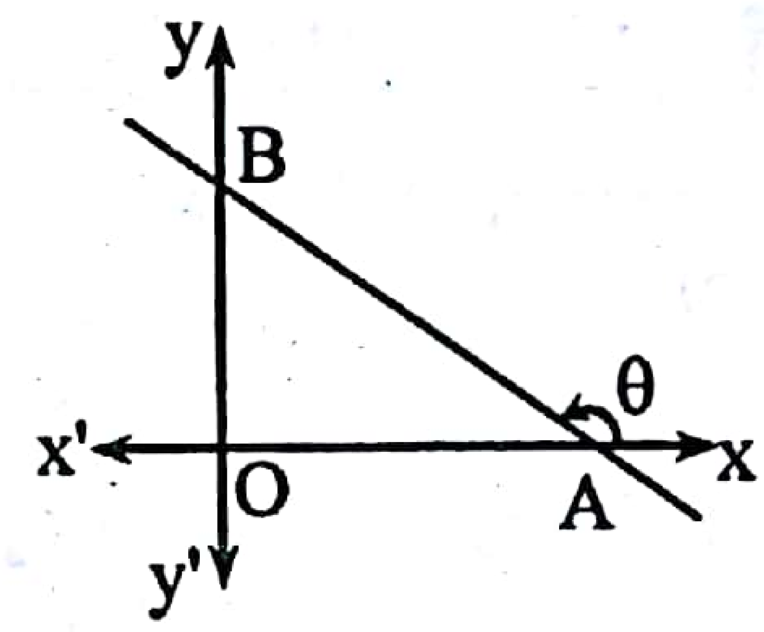
Note:

The combined equation of the coordinate axis is $xy = 0$

14. Slope of a line

Slope of a line means the inclination of the line with the positive direction of x-axis. Thus the trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called **slope or gradient** of a line denoted by (m). Let the line AB make θ angle with the x-axis then Gradient of line $AB = m = \tan\theta$





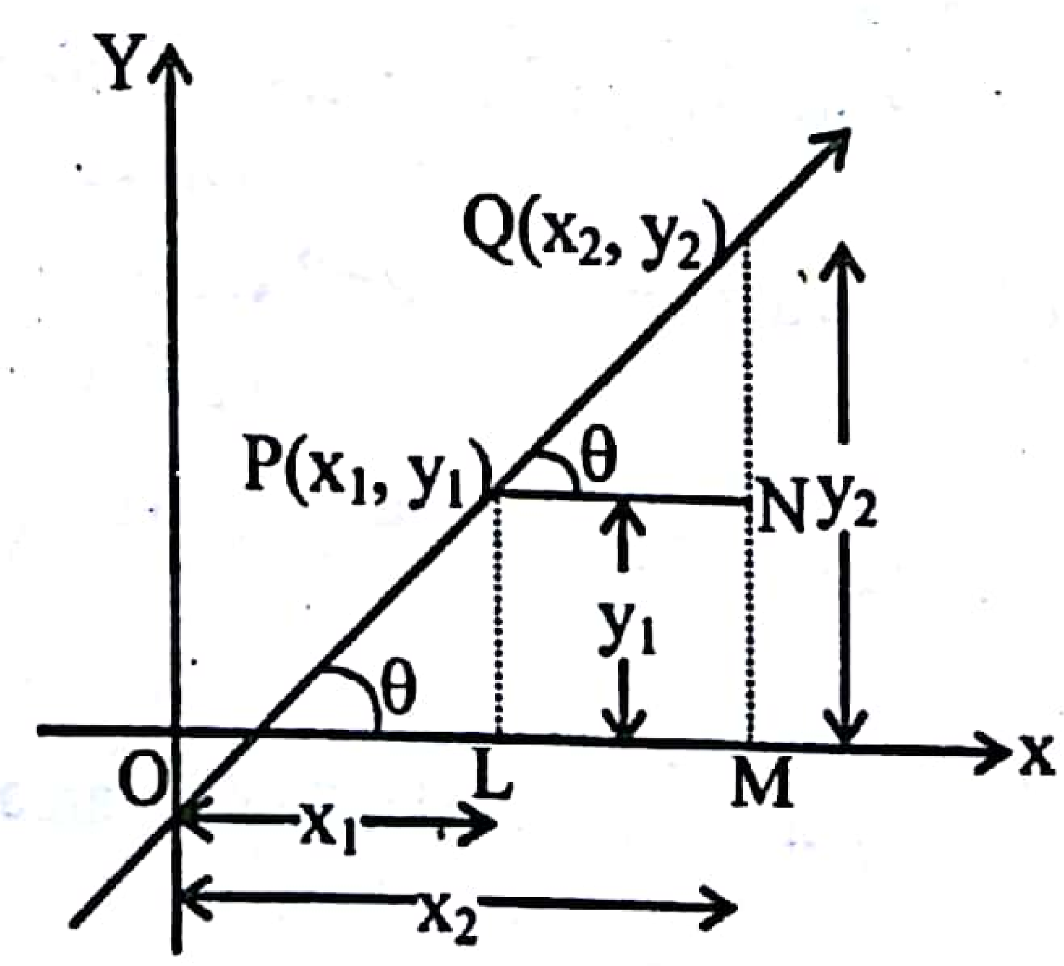
Note :

(i) Slope of x-axis or a line parallel to x-axis is Zero.

(ii) Slope of y-axis or a line parallel to y-axis is $m = \tan 90° = \infty$

14.1 Slope of a line in term of coordinates of any two point on it :

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be two point on a line making an angle θ with positive direction of x-axis.



In ΔPQN, $\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus slope of a line is

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Diff. of ordinate}}{\text{Diff. of abscissa}} = \frac{\Delta y}{\Delta x}$$

15. Different forms of the equation of straight line

15.1 Slope Intercept form :

The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$

If the line passes through the origin, then $c = 0$. Thus the equation of a line with slope m and passing through the origin $y = mx$.

15.2 Two points form equation of line :

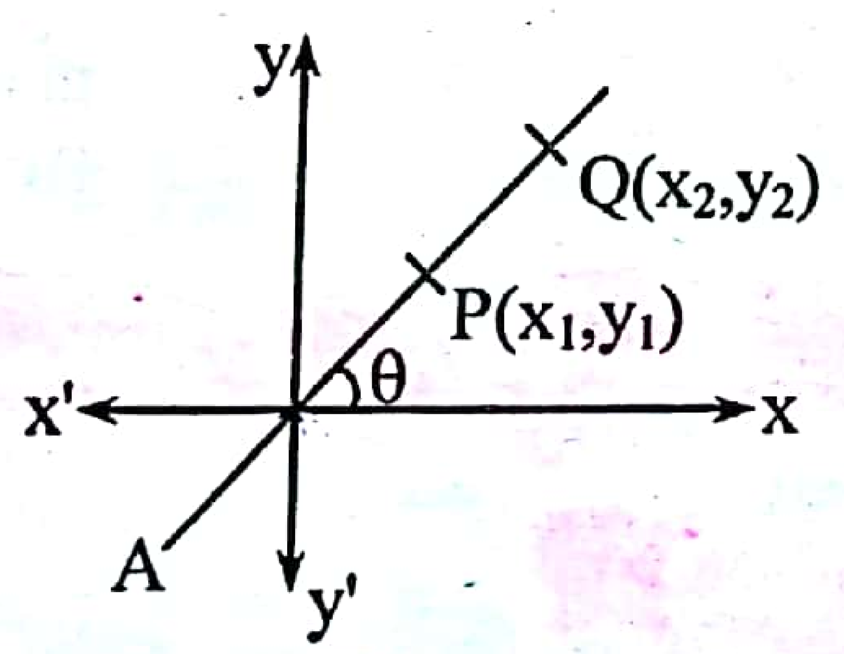
Let m be the slope of the line passing through (x_1, y_1) & (x_2, y_2). Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So the equation of the line is

$$y - y_1 = m(x - x_1) \quad ...(1)$$

putting the value of m in eq. (1)



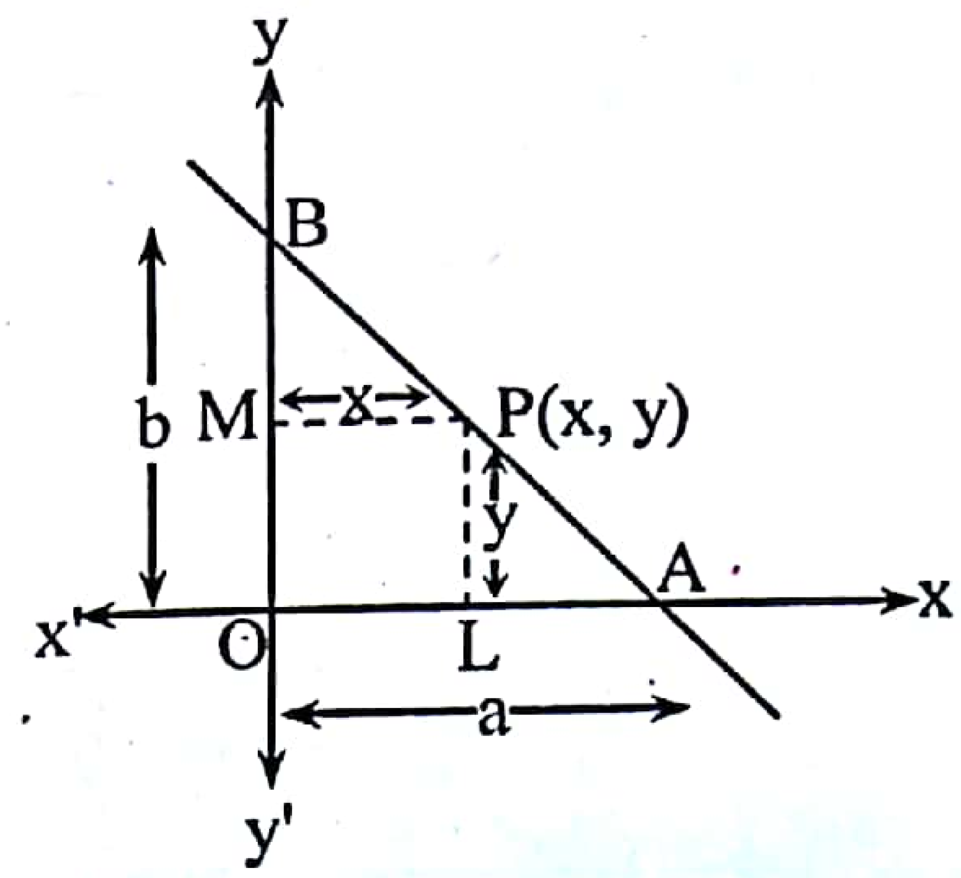
$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two points.

15.3 Intercept form of a line :

Let AB be the line which cuts off intercepts OA = a and OB = b on the x and y-axis and Let P(x, y) be any point on the line

Area of ΔOAB = Area of ΔOPA + Area of ΔOPB



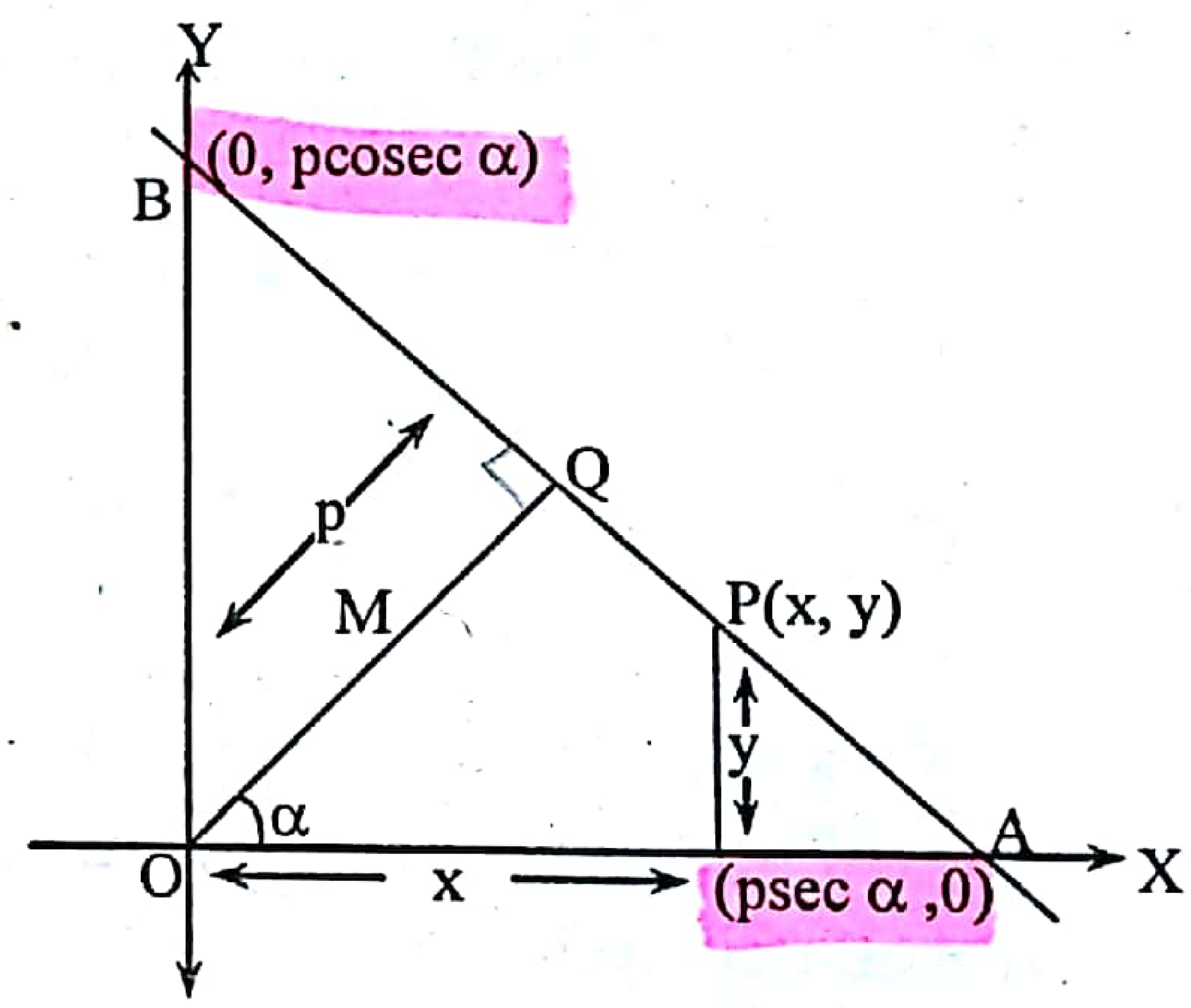
$$\therefore \frac{x}{a} + \frac{y}{b} = 1 \, (a \neq 0, b \neq 0)$$

This is the equation of line in the intercept form.

15.4 Normal (Perpendicular) form of a line :

Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be P and $\angle XOQ = \alpha$.

by using intercept form of line we get



$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\mathrm{cosec}\,\alpha} = 1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p \; (p > 0) \text{ and } \alpha \in [0, 2\pi]$$

which is the required equation of line

15.5 Parametric or distance form of a line :

Let the given line meets x-axis at A and y-axis at B and passes through the point $Q(x_1, y_1)$. Let P(x, y) be any point on the line at a distance r from $Q(x_1, y_1)$. Then

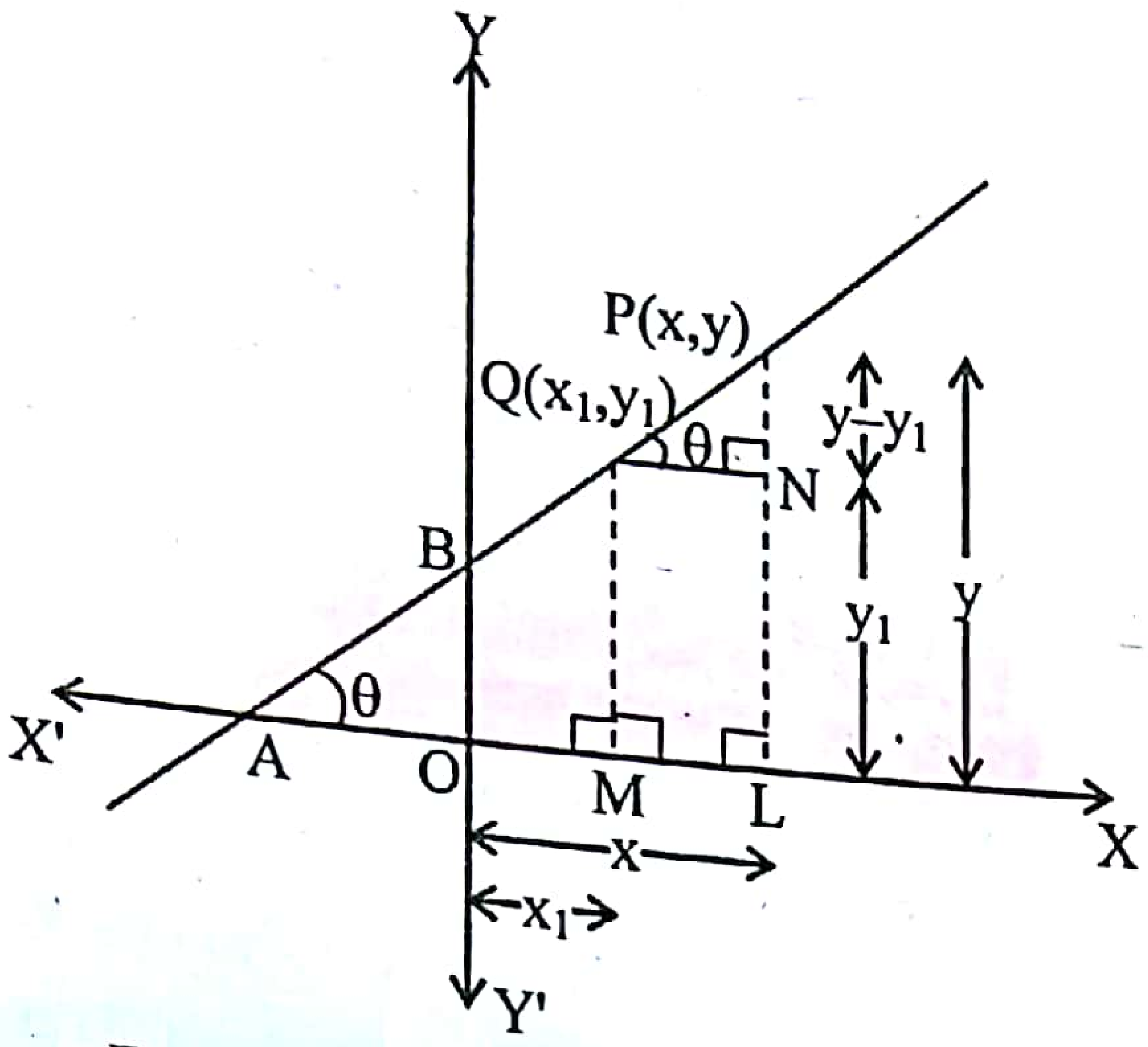
$QN = ML = OL - OM = x - x_1$

and $PN = PL - NL = y - y_1$

From ΔPQN, $\cos\theta = \frac{QN}{PQ} = \frac{x - x_1}{r}$

$$\Rightarrow \cos\theta = \frac{x - x_1}{r} \quad ...(i)$$

Again $\sin\theta = \frac{PN}{PQ} \Rightarrow \sin\theta = \frac{y - y_1}{r}$ ...(ii)



∴ From (i) & (ii) we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the equation of the line in the distance form.

Note:

(i) The equation of the line is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = \pm r$ where $r \in [0, \infty)$

$\therefore x - x_1 = \pm r\cos\theta$ and $y - y_1 = \pm r\sin\theta$

$\Rightarrow x = x_1 \pm r\cos\theta$ and $y = y_1 \pm r\sin\theta$

Thus the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.

(ii) If θ is variable and x_1, y_1 and r are fixed then the same equation will represent a circle whose centre is (x_1, y_1) and radius is r.

(iii) This form is selected where the portion of a line or line segment is to be discussed.

15.6 Determinant form equation of line :

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ are the two given point then equation of the line passing through these is given by $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

This can also be written in another form.

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

15.8 General form : We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and

intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

16. Angle between two lines

(a) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

Note :

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of tanθ.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these lines are given by,

$\tan A = \frac{m_1 - m_2}{1 + m_1m_2}$; $\tan B = \frac{m_2 - m_3}{1 + m_2m_3}$ &

$\tan C = \frac{m_3 - m_1}{1 + m_3m_1}$

(b) **Condition for perpendicular lines:**

If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of 90°

$\tan 90° = \frac{m_2 - m_1}{1 + m_2m_1} \Rightarrow m_1m_2 = -1$

Thus when two lines are perpendicular the product of their slopes is – 1. If m is the slope of a line then the slope of a line perpendicular to it is $-\frac{1}{m}$.

Note:

(i) Lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are parallel, if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

(ii) Lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are perpendicular, if $a_1a_2 + b_1b_2 = 0$

(c) Condition for Coincident lines :

Two lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are coincident only and only if.

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

17. Equation of lines parallel and perpendicular to a given line

(a) Equation of line parallel to line $ax + by + c = 0$ is $ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$ is $bx - ay + k = 0$.

Here λ, k are parameters and their values are obtained with the help of additional information given in the problem.

Note:

To write a line perpendicular to a given line following steps are taken

Step I : Interchange x and y.

Step II : If the coefficients of x and y in the given equation are of the same sign make them of opposite signs and if the coefficients are of opposite signs make them of the same sign.

Step III : Replace the given constant by a new constant which is determined by a given condition.

18. Straight line making a given angle with a line

Equations of lines passing through a point (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

19. Length of perpendicular from a point on a line

Length of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is $p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is

$$p = \frac{|c|}{\sqrt{a^2 + b^2}}$$

20. Distance between two parallel lines

(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

Note :

The coefficients of x & y in both equations should be same.

(b) The area of the parallelogram $= \frac{p_1p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

To find the distance between these two lines following steps are taken

Step I : Choose a point on any one of the line by giving a particular value of x or y.

Step II : Then find the length of the perpendicular from the point to line. The length so obtained is the required distance between the parallel lines.

21. Position of two points with respect to a given line

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

22. Intersection of two lines

Let the equations of two lines be

$a_1x + b_1y + c_1 = 0$ ...(i)

and $a_2x + b_2y + c_2 = 0$ ...(ii)

Let these two lines intersect at a point P (x_1, y_1). Then (x_1, y_1) satisfies each of the given equations

$\therefore\ a_1x_1 + b_1y_1 + c_1 = 0$ and

$a_2x_1 + b_2y_1 + c_2 = 0$

Solving these two equations

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Hence the coordinates of the point of intersection of (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

23. Concurrency of lines

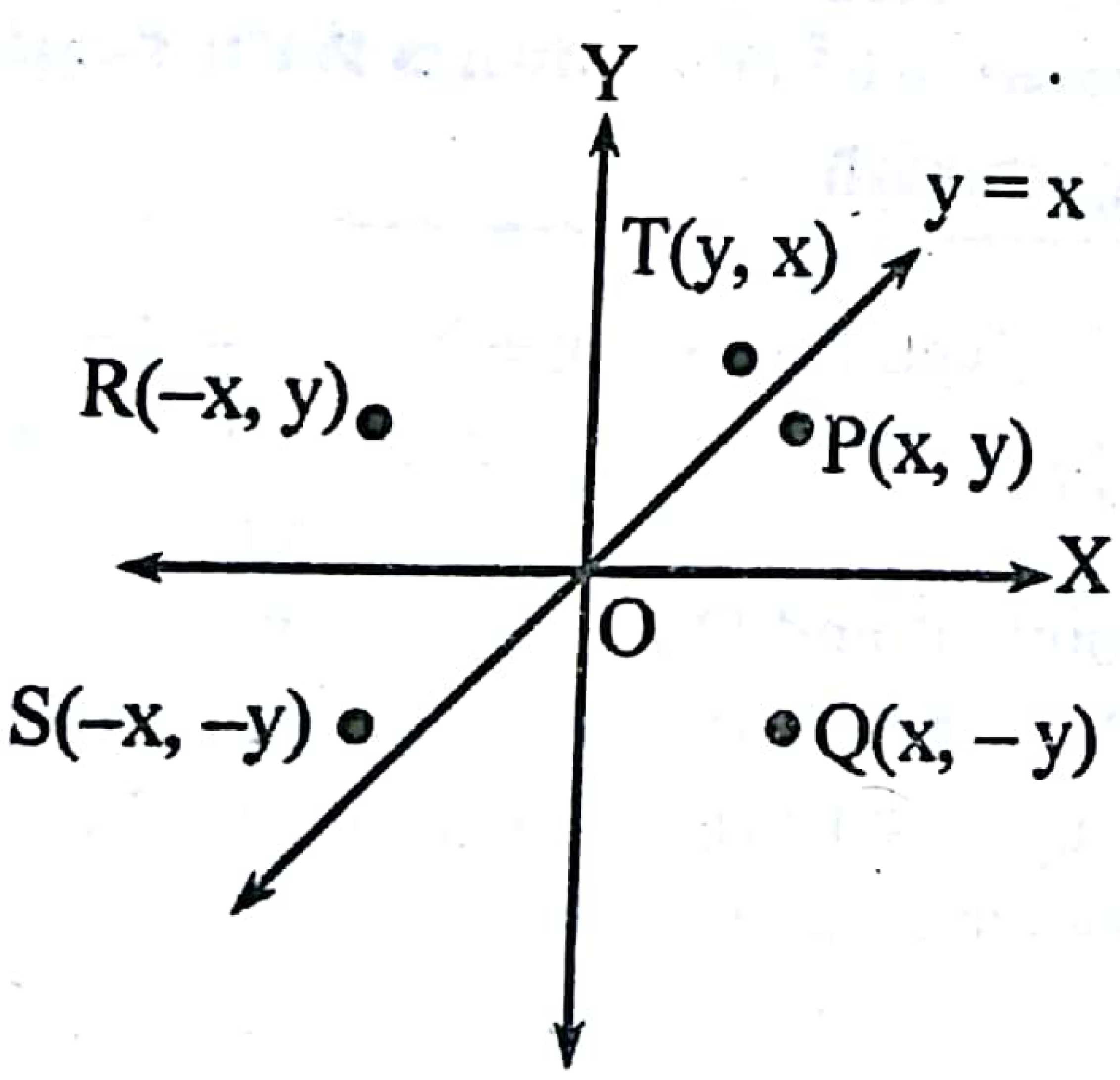
(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

24. Reflection of a point

Let P(x, y) be any point, then its image with respect to



(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x, –y)

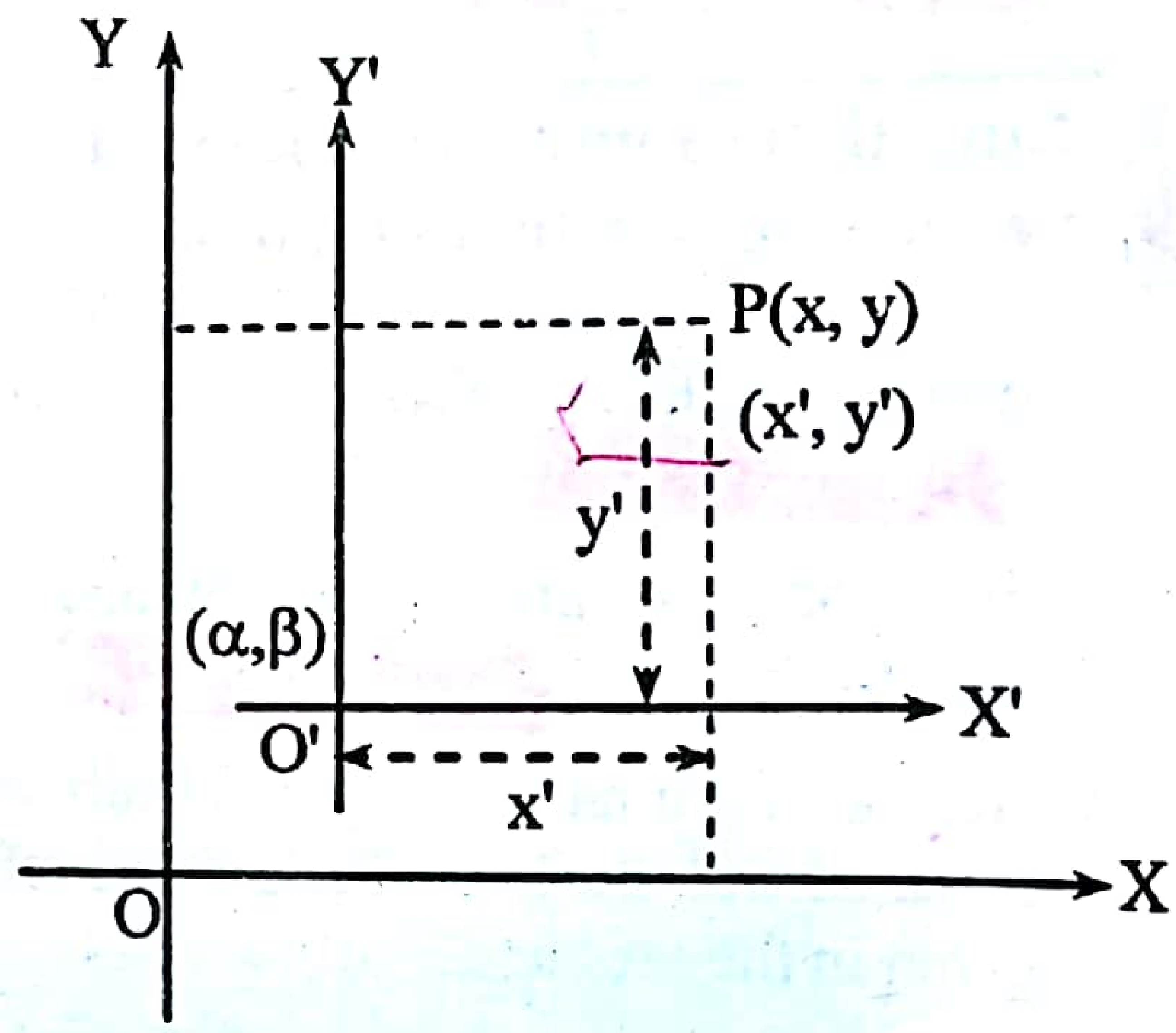
(d) line y = x is T(y, x)

25. Transformation of axes

(a) Shifting of origin without rotation of axes :

Let P ≡ (x, y) with respect to axes OX and OY.

Let O' ≡ (α, β) is origin with respect to axes OX and OY and let P ≡ (x', y') with respect to axes O'X' and O'Y', where OX and O'X' are parallel and OY and O'Y' are parallel.



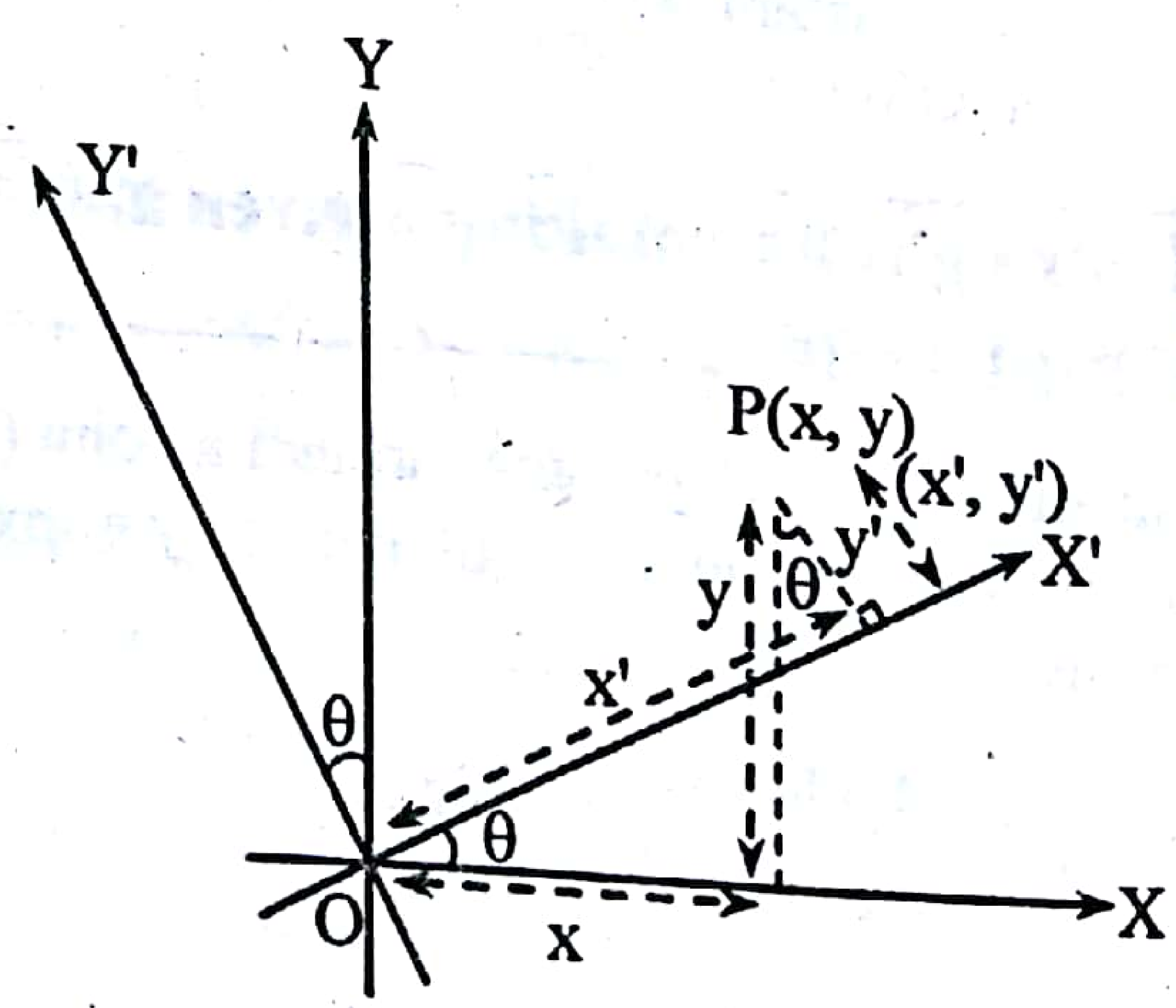
then $x = x' + \alpha$, $y = y' + \beta$

or $x' = x - \alpha$, $y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting x + α in place of x and y + β in place of y.

(b) **Rotation of axes without shifting the origin :**

Let O be the origin. Let P ≡ (x, y) with respect to axes OX and OY and let P ≡ (x', y') with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$



then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	x ↓	y ↓
x' →	$\cos\theta$	$\sin\theta$
y' →	$-\sin\theta$	$\cos\theta$

26. The equation of a Family of straight lines passing through the points of intersection of two given lines

The equation of a family of lines passing through the point of intersection of $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ is given by $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.

Note :

If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$ then, $u_1 = 0$; $u_2 = 0$; $u_3 = 0_3$; $u_4 = 0$ form a parallelogram.

$u_2u_3 - u_1u_4 = 0$ represents the diagonal BD.

Proof :

Since it is the first degree equation in x & y it is a straight line. Secondly point B satisfies the equation because the co-ordinates of B satisfy $u_2 = 0$ and $u_1 = 0$.

Similarly for the point D. Hence the result.

On the similar lines $u_1u_2 - u_3u_4 = 0$ represents the diagonal AC.

Note :

The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some λ and μ.

[For getting the values of λ & μ compare the coefficients of x, y & the constant terms].

27. Bisectors of the angles between two lines

(i) Equations of the bisectors of angles between the lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

(ii) **To discriminate between the acute angle bisector & the obtuse angle bisector**

If θ be the angle between one of the lines & one of the bisectors, find $\tan\theta$.

If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.

If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector.

(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then;

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ given the equation of the bisector of the angle containing the origin & $\frac{ax+by+c}{\sqrt{a^2+b^2}} = - \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin.

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows write $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that constant terms are positive. If $aa' + bb' < 0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ therefore

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = - \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ is the equation

of other bisector. If, however, $aa' + bb' > 0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}};\ \text{therefore}$$

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$ is the equation of other bisector.

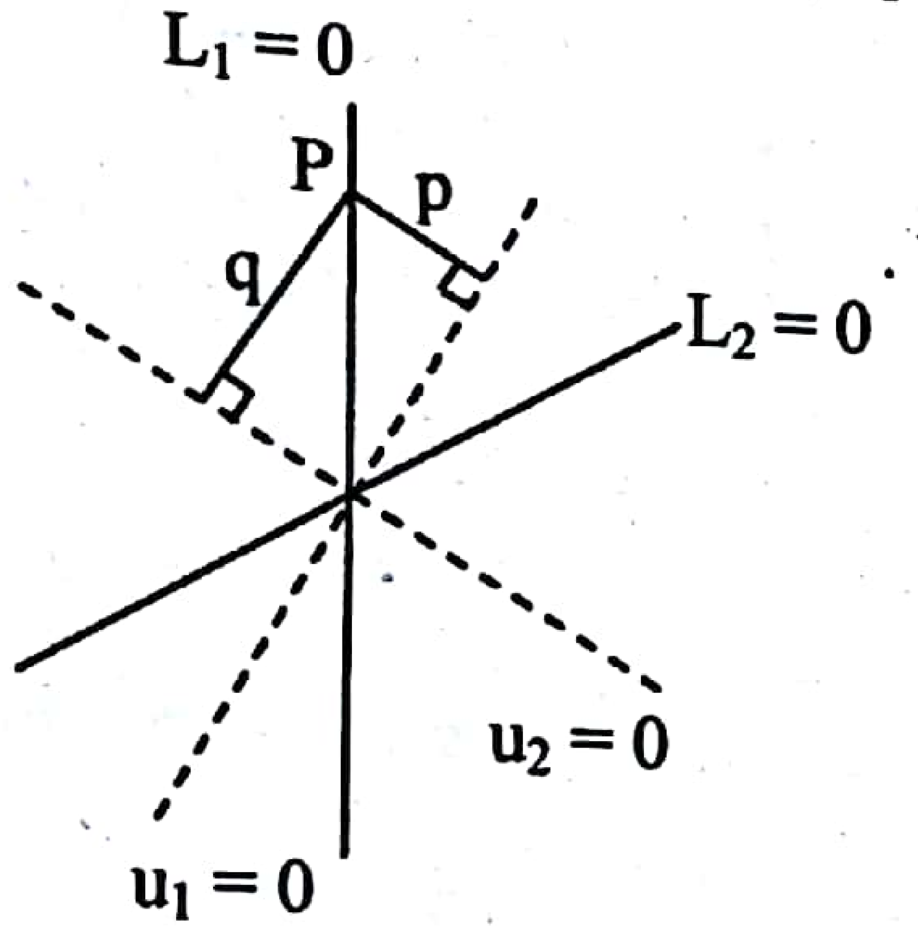
(v) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown. If,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.



Note :

Equation of straight lines passing, through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

28. A pair of straight lines through origin

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by, $ax^2+2hxy+by^2=0$,

then; $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$.

The condition that these lines are :

(a) At right angles to each other is $a + b = 0$. i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$ i.e. coeff. of $xy = 0$.

Note :

A homogeneous equation of degree n represents n straight lines passing through origin.

29. General equation of second degree representing a pair of straight lines

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

(iii) If θ be the angle between the lines, then

$\tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are

(a) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(b) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$

Note :

(a) The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by

$\ell x + my + n = 0$ ....(i)

& the 2nd degree curve:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(ii) is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) +$

$$2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0 \quad ...(iii)$$

(iii) is obtained by homogenizing (ii) with the help of (i), by writing (i) in the form : $\left(\frac{\ell x + my}{-n}\right) = 1$

(b) The equation to the straight lines bisecting the angle between the straight lines,

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

(c) The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

(d) Any second degree curve through the four point of intersection of f(xy) = 0 & xy = 0 is given by f(xy) + λxy = 0 where f(xy) = 0 is also a second degree curve.

(e) A homogeneous equation of degree n represents n straight lines passing through origin.

(f) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2^{nd} degree is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, $a \neq b, h \neq 0$.

(i) If $a = b$, the bisectors are $x^2 - y^2 = 0$

i.e. $x - y = 0, x + y = 0$

(ii) If $h = 0$, the bisectors are $xy = 0$

i.e. $x = 0, y = 0$

(iii) The two bisectors are always at right angles, since we have

coeff. of x^2 + coeff. of $y^2 = 0$

(g) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

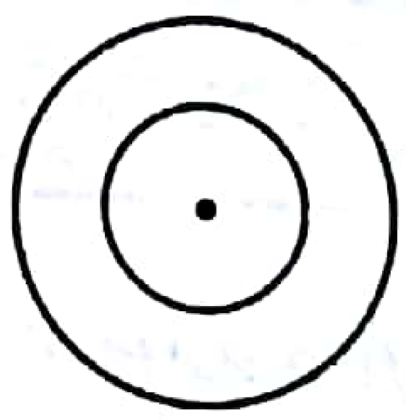
KEY CONCEPTS

1. Definition

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

2. Basic theorems & Results of circles

(a) **Concentric circles :** Circles having same centre.



(b) **Congruent circles :** Iff their radii are equal.

(c) **Congruent arcs :** Iff they have same degree measure at the centre.

Theorem 1 :

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.

Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Converse : If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2 :

(i) The perpendicular from the centre of a circle to a chord bisects the chord.

Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3 :

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4 :

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.

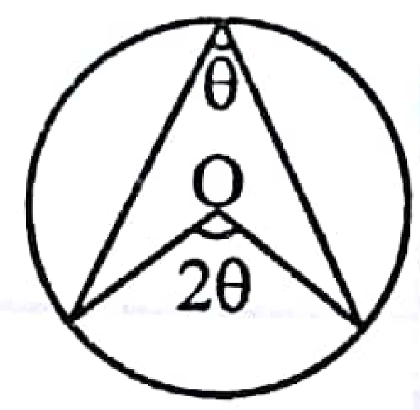
Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal.

(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

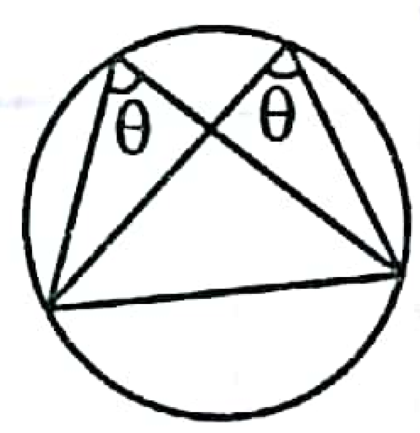
(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5 :

(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

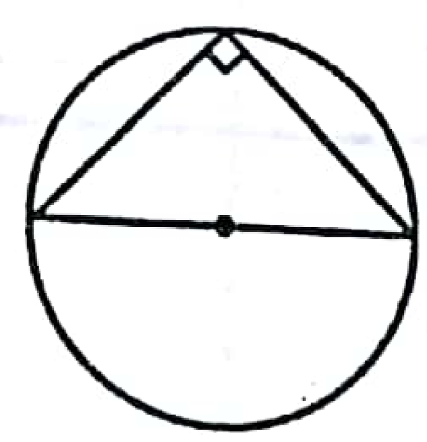


(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.

Converse : The arc of a circle subtending a right angle in alternate segment is semi circle .



Theorem 6 :

Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7 :

If a line segment joining two point subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals :

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1 :

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

or

The opposite angles of a cyclic quadrilateral are supplementary.

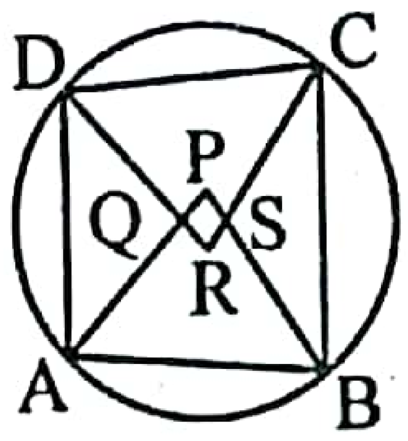
Converse : If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2 :

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Theorem 3:

The quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.



Theorem 4 :

If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal.

or

A cyclic trapezium is isosceles and its diagonals are equal.

Converse : If two non-parallel sides of a trapezium are equal, then it is cyclic,

or

An isosceles trapezium is always cyclic.

Theorem 5 :

The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided that they are not parallel), intersect at right angle.

3. Tangents to a circle

Theorem 1 :

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

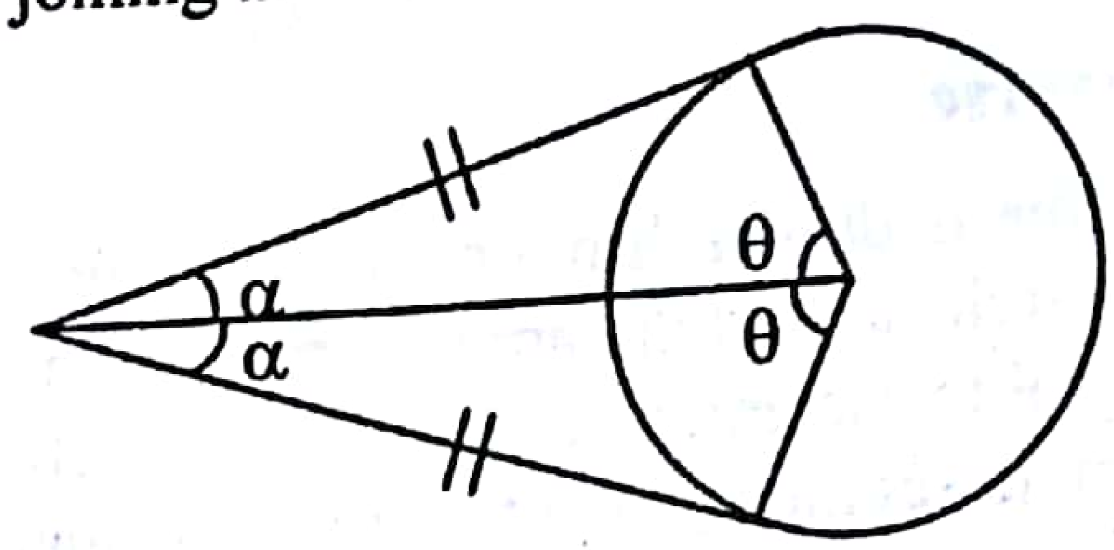
Theorem 2 :

If two tangents are drawn to a circle from an external point, then :

(i) they are equal.

(ii) they subtend equal angles at the centre,

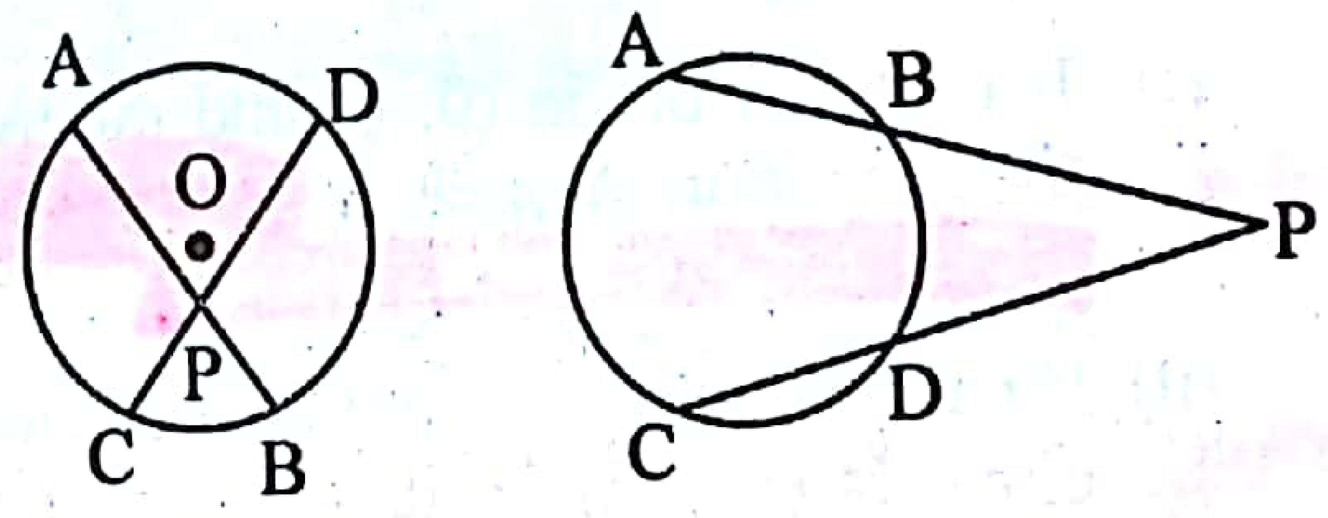
(iii) they are equally inclined to the segment, joining the centre to that point.



Theorem 3 :

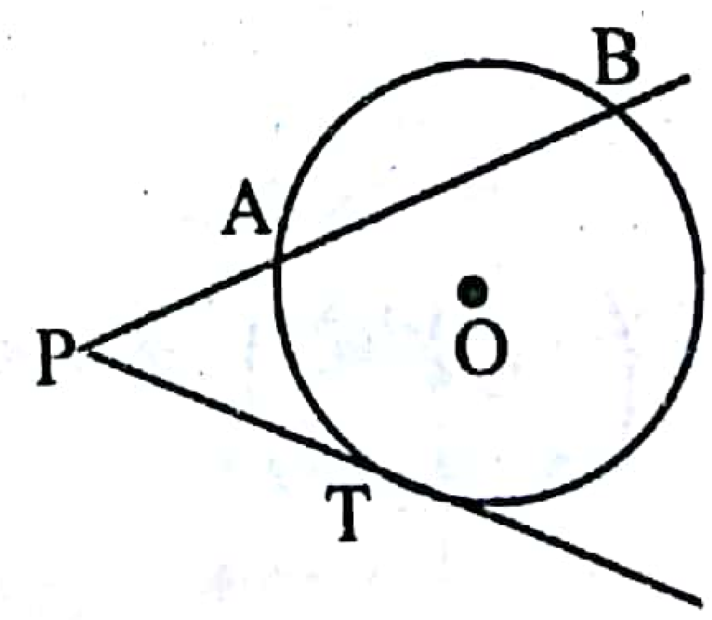
If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord.

$PA \times PB = PC \times PD$



Theorem 4 :

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then $PA \times PB = PT^2$



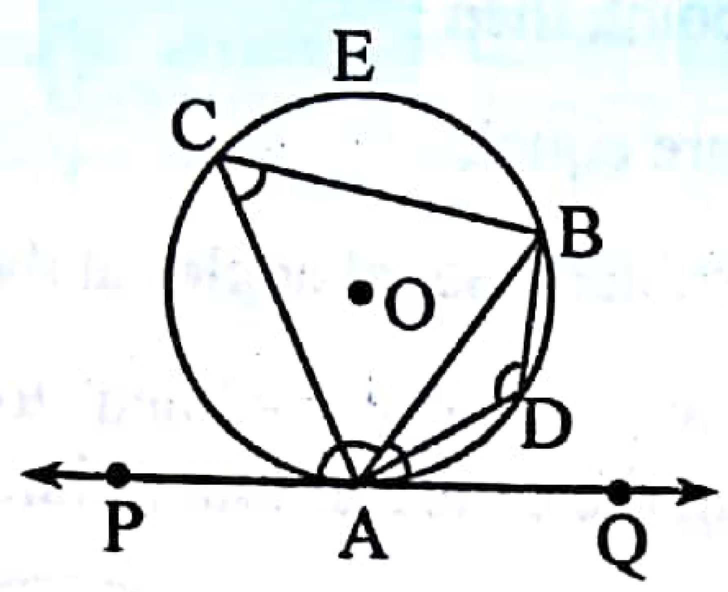
or

Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 5 :

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$



Converse :

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

4. Standard equations of the circle

(a) Central form :

If (h, k) is the centre and r is the radius of the circle then its equation is

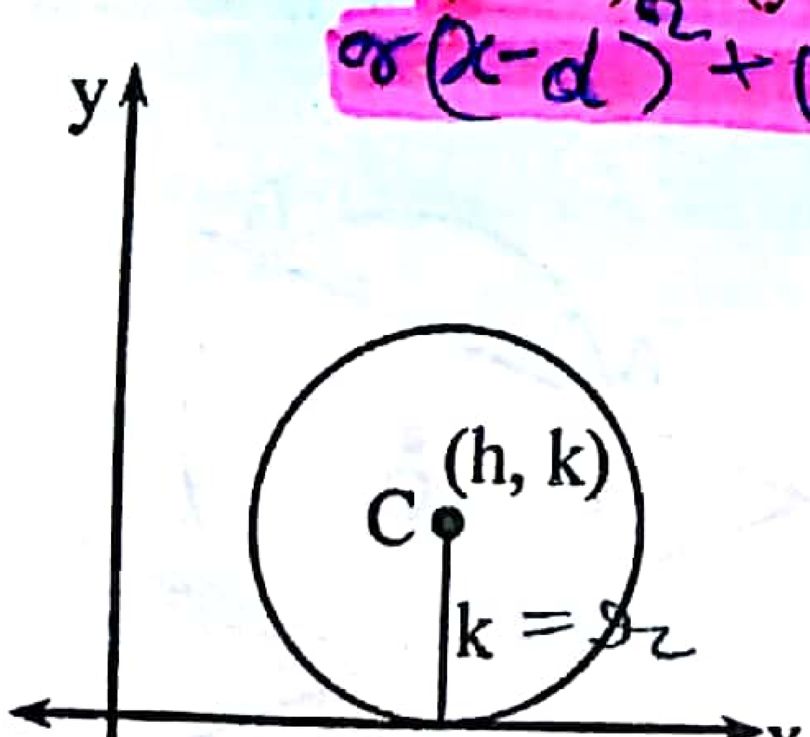
$(x-h)^2 - (y-k)^2 = r^2$

Special Cases :

(i) If centre is origin (0, 0) and radius is 'r' then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $(x-h)^2 + (y-k)^2 = 0$. Such circle is called zero circle or point circle.

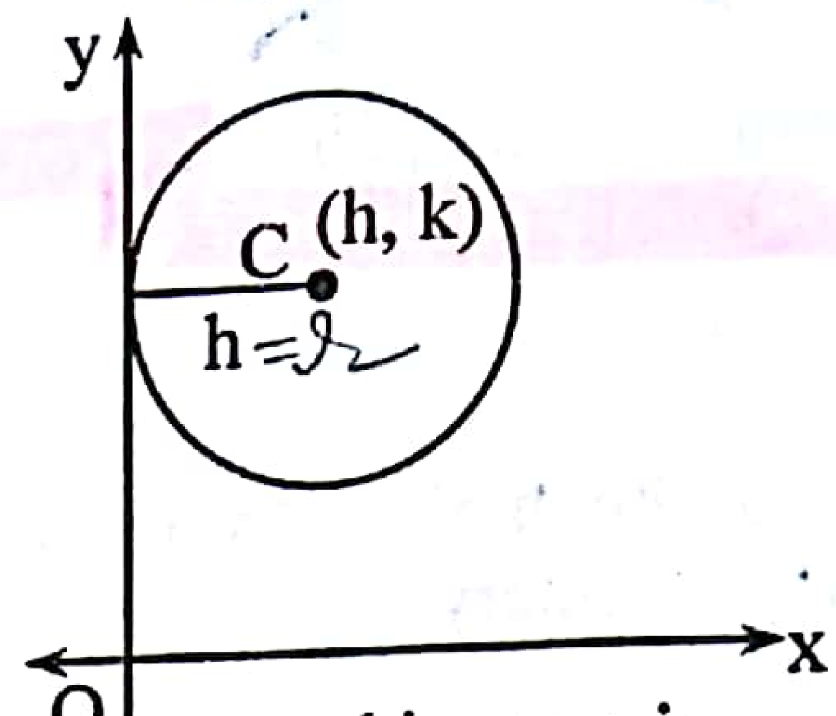
(iii) When circle touches x-axis then equation of the circle is $(x-h)^2 + (y-k)^2 = k^2$



Touching x-axis

(iv) When circle touches y-axis then equation of circle is $(x-h)^2 + (y-k)^2 = h^2$

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$

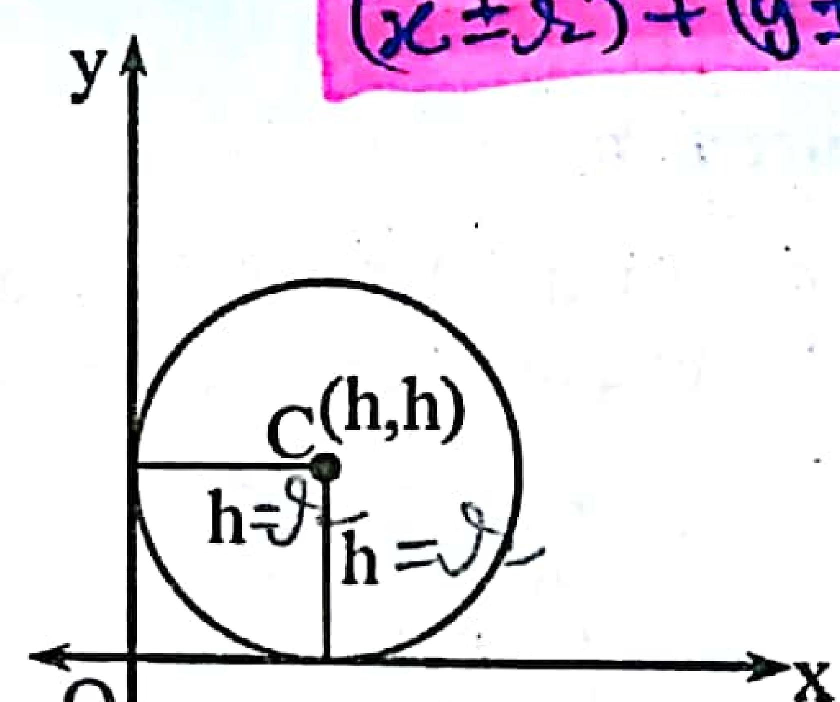


Touching y-axis

(v) When circle touches both the axes (x-axis and y-axis) then equation of circle

$(x-h)^2 + (y-h)^2 = h^2$

or $x^2 + y^2 - 2hx - 2hy + h^2 = 0$

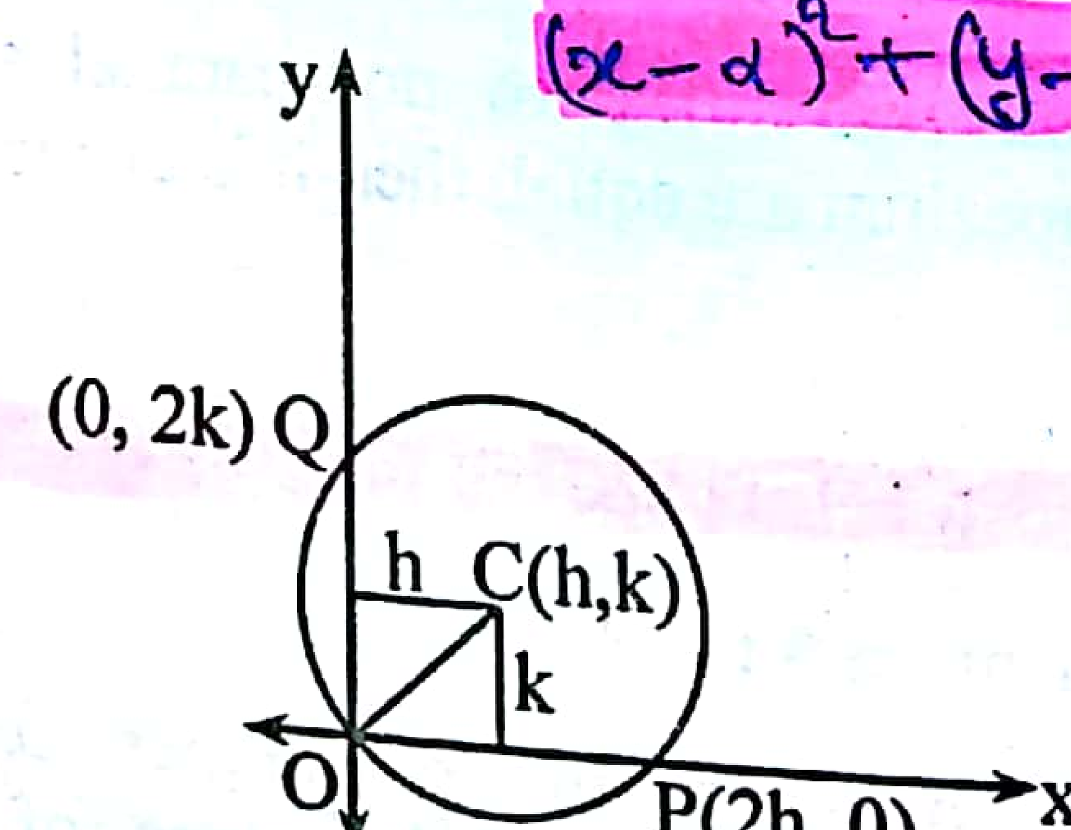


Touching x-axis and y-axis

(vi) When circle passes through the origin and centre of the circle is (h, k) then radius $\sqrt{h^2 + k^2} = r$ and intercept cut on x-axis OP = 2h, and intercept cut on y-axis is OQ = 2k and equation of circle is

$(x-h)^2 + (y-k)^2 = h^2 + k^2$

or $x^2 + y^2 - 2hx - 2ky = 0$



Note :

Circle may exist in any quadrant hence for general cases use ± sign before h & k.

(b) General equation of circle :

$x^2 + y^2 + 2gx + 2fy + c = 0$ where g,f,c are constants and centre is $(-g, -f)$

i.e. $\left(-\frac{\text{coeff. of } x}{2}, -\frac{\text{coeff. of } y}{2}\right)$ and

radius $r = \sqrt{g^2 + f^2 - c}$

Note :

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius $r = 0$ and circle is a point circle .

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary then circle is also an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non collinear points.

(v) The general quadratic equation in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of xy = 0 or h = 0
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ on:

(i) x-axis $= 2\sqrt{g^2 - c}$ (x का उल्टा y से है।)

(ii) y-axis $= 2\sqrt{f^2 - c}$

Note :

(i) If the circle cuts the x-axis at two distinct point then $g^2 - c > 0$

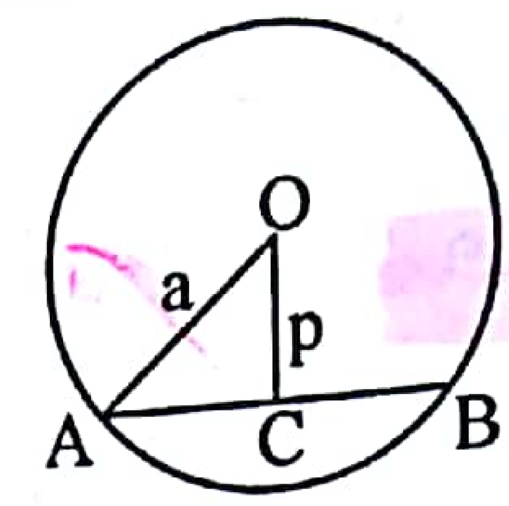
(ii) If circle touches x-axis then $g^2 = c$.

(iii) If circle touches y-axis then $f^2 = c$.

(iv) Circle lies completely above or below the x-axis then $g^2 < c$.

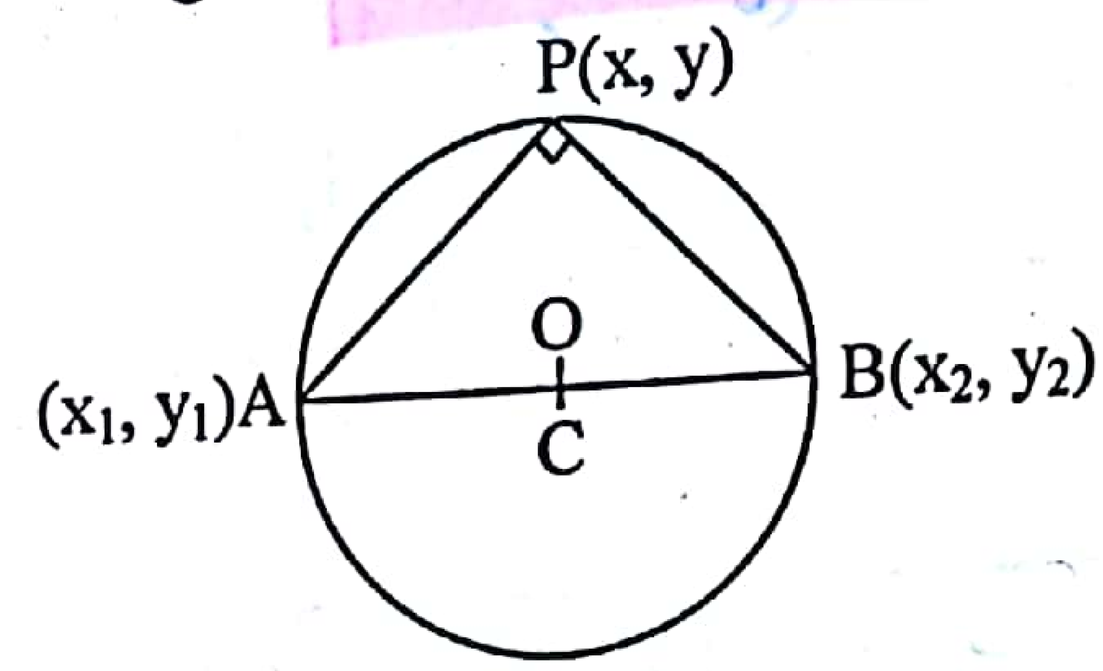
(v) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - p^2}$ where a is the radius and p is the length of perpendicular from the centre to the chord.

Length of chord $= 2\sqrt{a^2 - p^2}$



(d) Diameter form of circle :

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of the diameter of the circle and P(x, y) is the point other then A and B on the circle then from geometry we know that $\angle APB = 90°$



$\Rightarrow$ (Slope of PA) × (Slope of PB) = – 1

$\Rightarrow \therefore \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Note :

This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2).

(e) The parametric forms of the circle :

(i) The parametric equation of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta, y = r\sin\theta; \theta \in [0, 2\pi)$

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$,

$y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$ where θ is parameter.

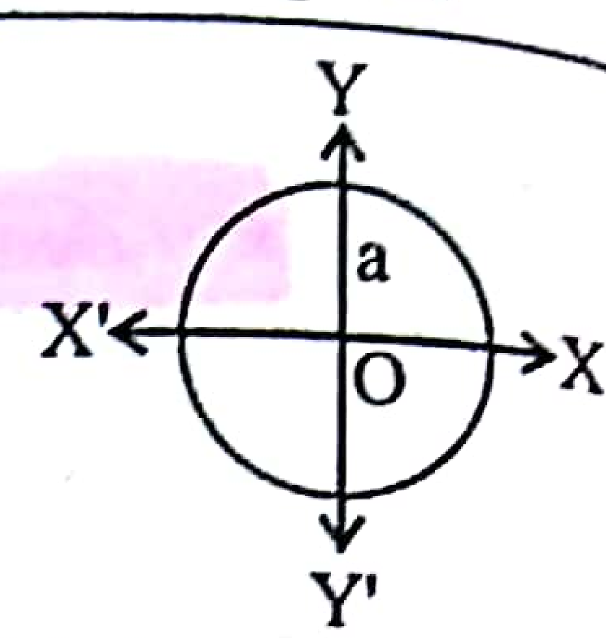
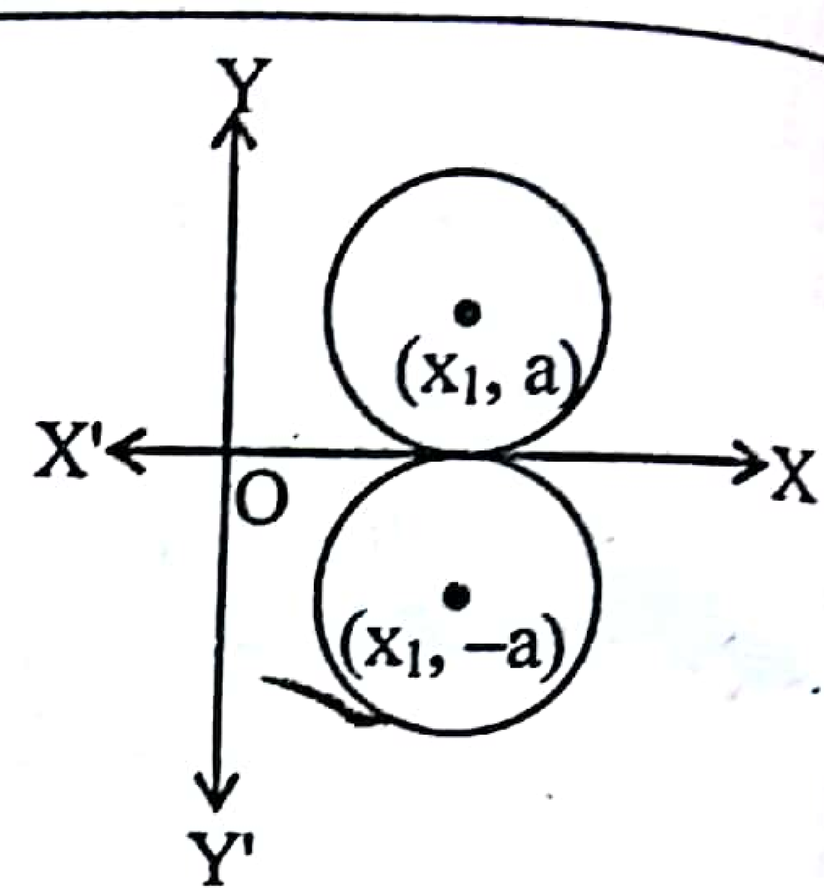
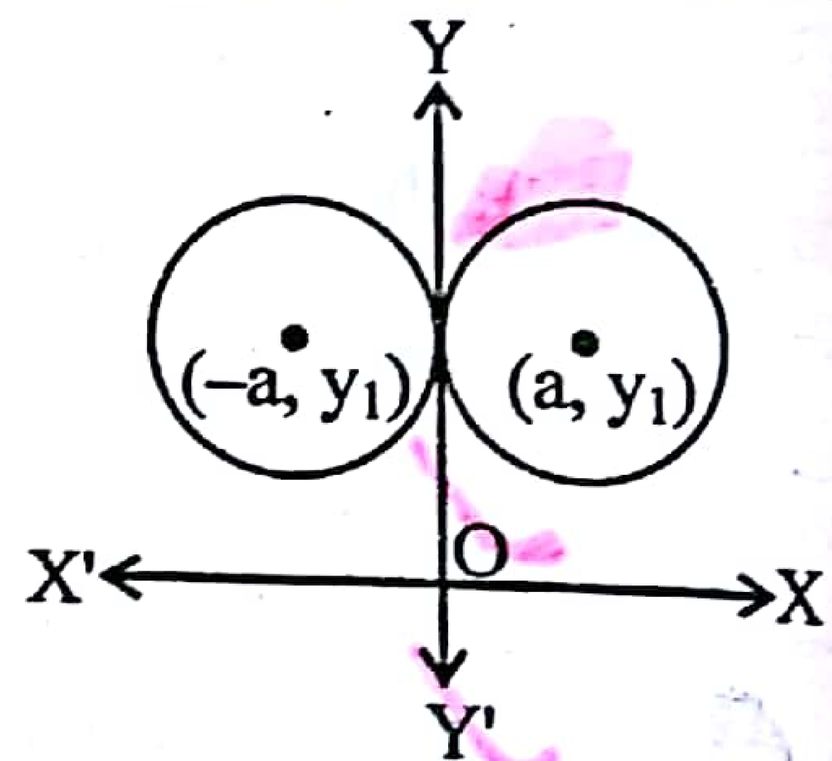
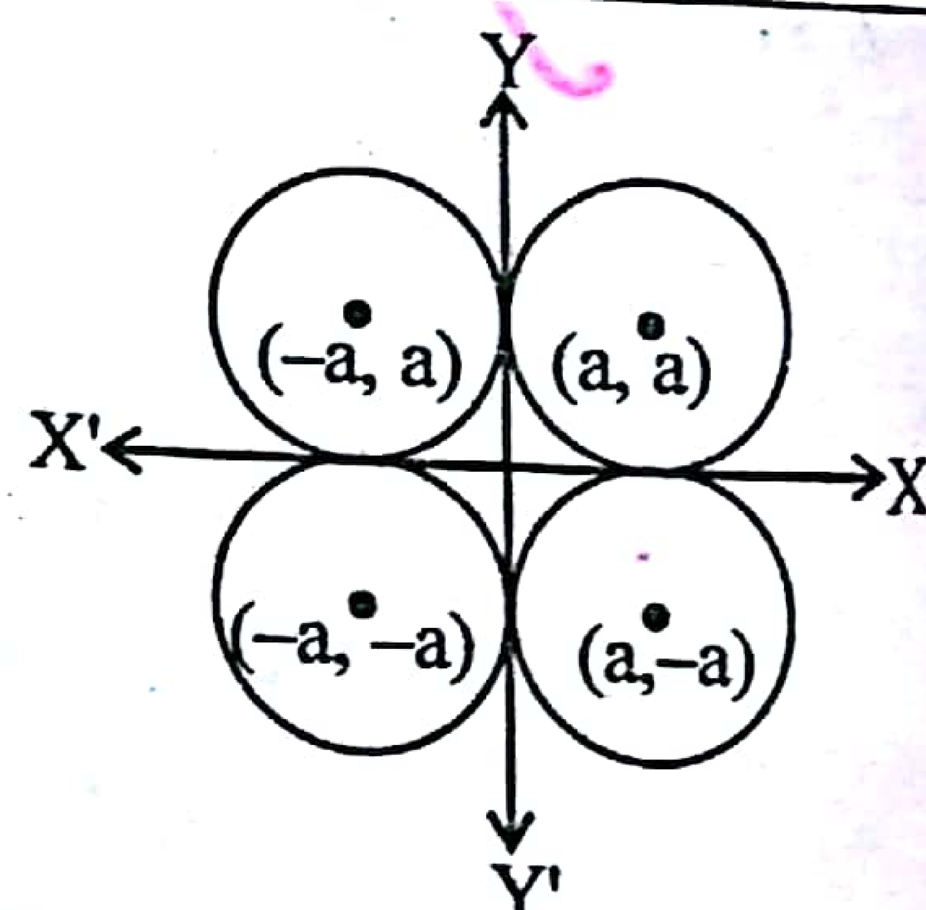
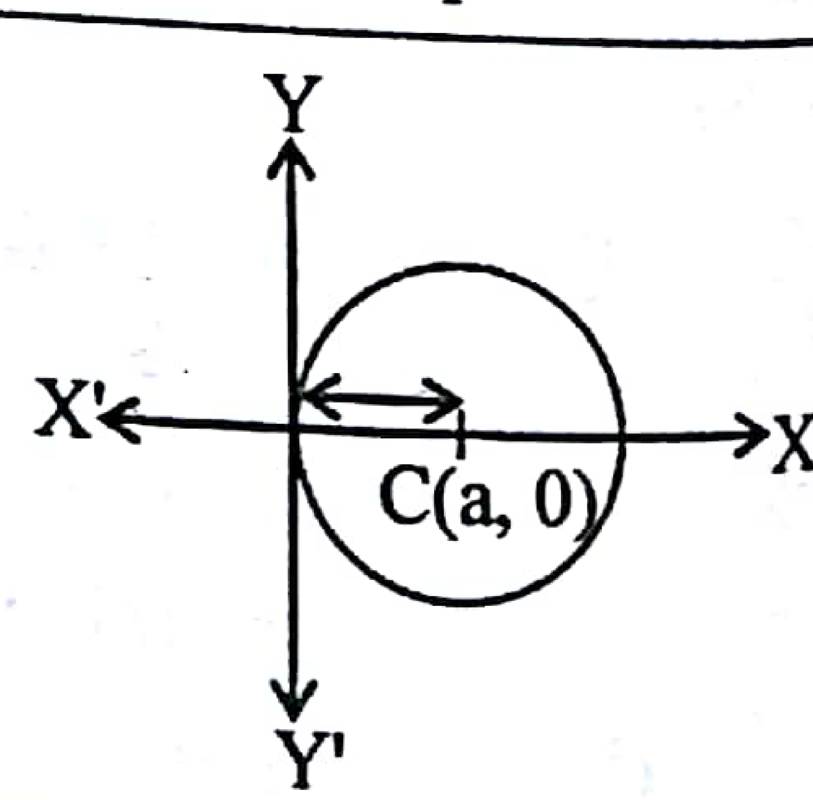
Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

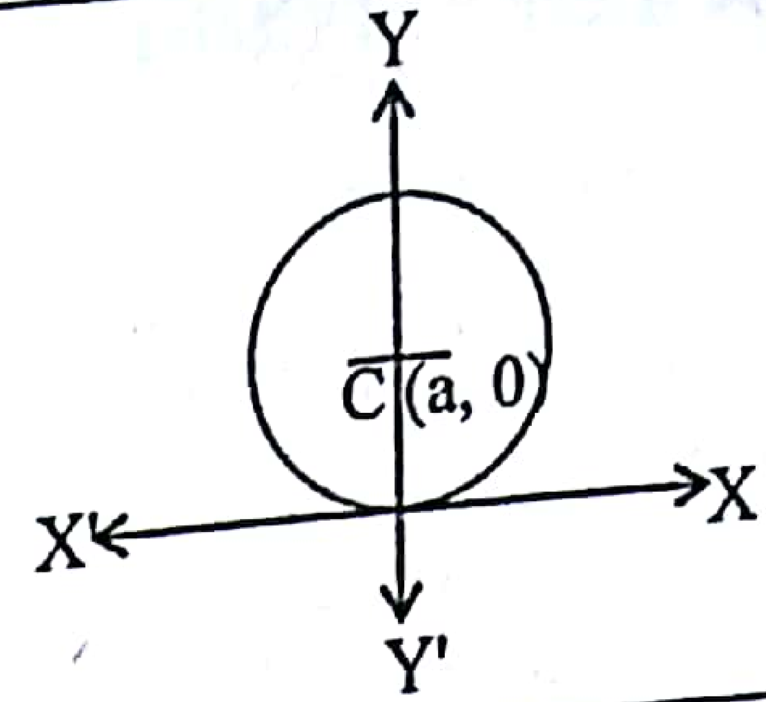
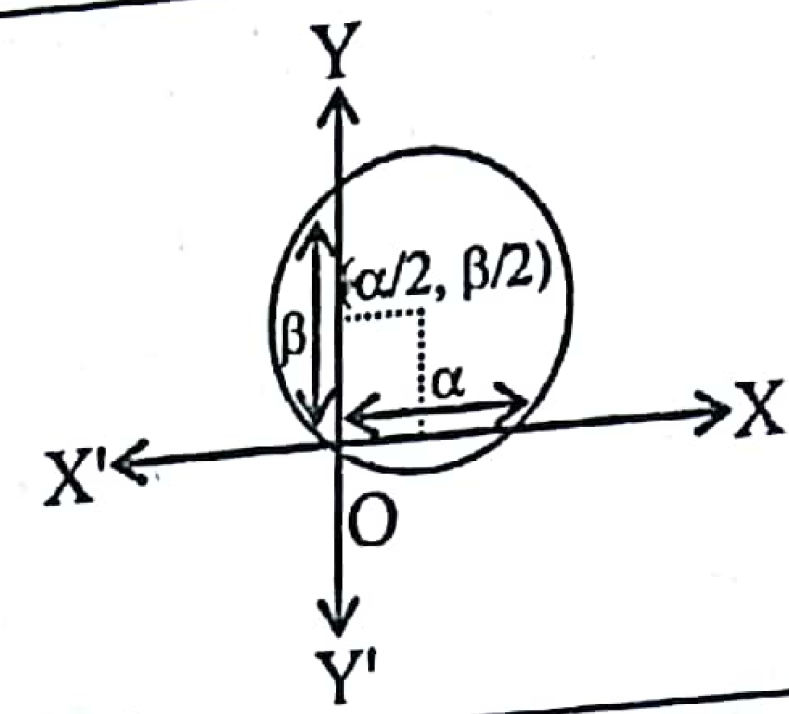
$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

Eqⁿ of chord joining two points.

5. Particular forms of equation of circle

The equation of a circle with centre at (h, k) and radius equal to a is $(x-h)^2+(y-k)^2=a^2$ .....(i)

Equation	Centre/ Radius	Properties	Figures
(a) $x^2+y^2=a^2$ $x^2+y^2=r^2$	(0, 0), a	When the centre of the circle coincides with the origin Centre = (0, 0)	
(b) $(x-x_1)^2+(y-a)^2=a^2$ $(x-x_1)^2+(y\pm r)^2=r^2$	(x_1, a) ; a	Touches x axis only, y coordinate of centre = ± a	
(c) $(x-a)^2+(y-y_1)^2=a^2$ $(x\pm r)^2+(y-y_1)^2=r^2$	(a, y_1) ; a	Touches y axis only, x coordinate of centre = ± a	
(d) $(x-a)^2+(y-a)^2=a^2$ $(x\pm r)^2+(y\pm r)^2=r^2$	(a, a) ;.a	Touches both the axes depending on the quadrant centre = (± a, ± a)	
(e) $x^2+y^2-2ax=0$ $x^2+y^2-2rx=0$ Centre जब x-axis में हो तब ये होगा	C (a, 0) ; a	When the circle passes through the origin and centre lies on x axis at (a, 0) point	

(f) $x^2+y^2-2ay=0$ $x^2+y^2-2ay=0$	C (0, a) ; a	When the circle passes through the origin and centre lies on y axis at (0, a) point	
(g) $x^2+y^2-\alpha x-\beta y=0$	$(\alpha/2, \beta/2)$; $\sqrt{\dfrac{\alpha^2+\beta^2}{4}}$	Passes through (0, 0) and has intercepts α and β on the axes	

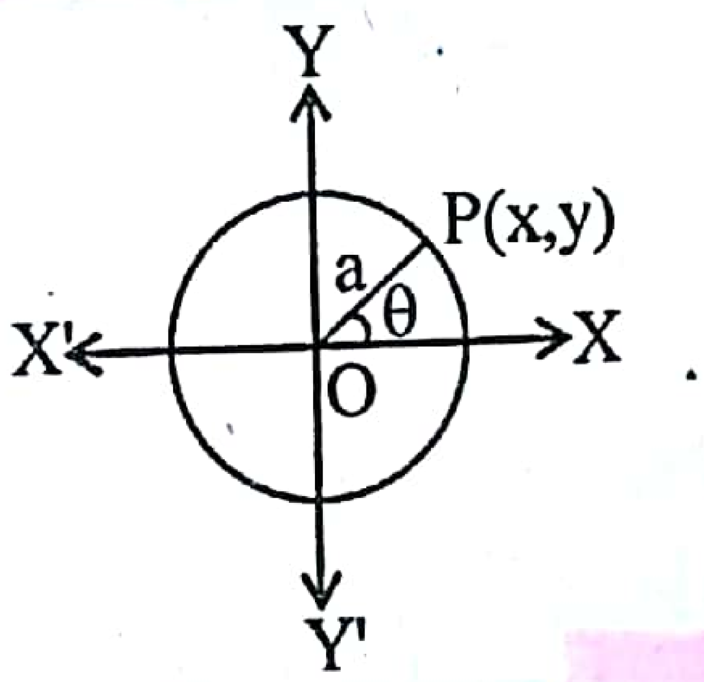
6. Parametric equation of a circle

(a) The parametric equation of

$x^2+y^2=a^2$ are

$x = a\cos\theta,\ y = a\sin\theta,$

where $-\pi<\theta\le\pi$ or $\theta\in[0, 2\pi)$

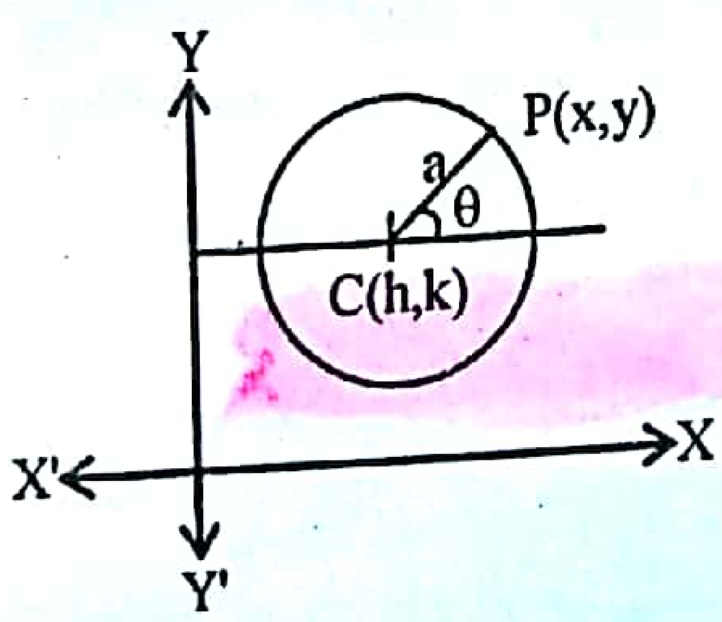


(b) The parametric equation of circle when the centre of circle is (h, k) and radius = a.

$(x-h)^2+(y-k)^2=a^2$ are

$x = h + a\cos\theta$; $y = k + a\sin\theta,$

where $-\pi<\theta\le\pi$ or $\theta\in[0, 2\pi)$



(c) The parametric equations of

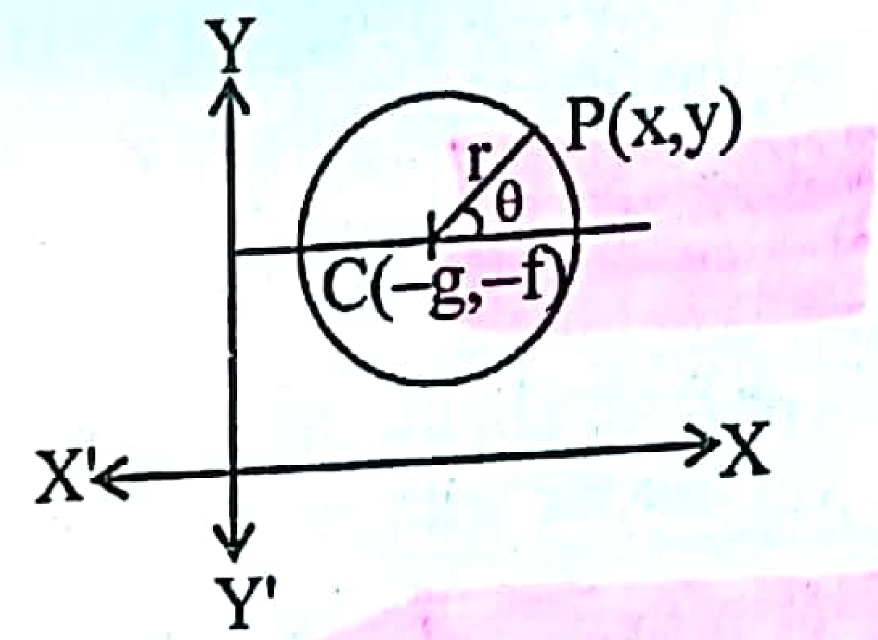
$x^2+y^2+2gx+2fy+c=0$ or

$(x+g)^2+(y+f)^2=g^2+f^2-c$ are

$x=-g+r\cos\theta,\ y=-f+r\sin\theta,$

where $-\pi<\theta\le\pi$ or $\theta\in[0, 2\pi)$ &

$r=\sqrt{g^2+f^2-c}$



7. Point Circle

The equation $(x-h)^2+(y-k)^2=0$ has its locus only the point (h, k), such a locus is called a **Point-circle** with centre (h, k) and radius equal to zero.

8. Position of a point w.r.t. circle

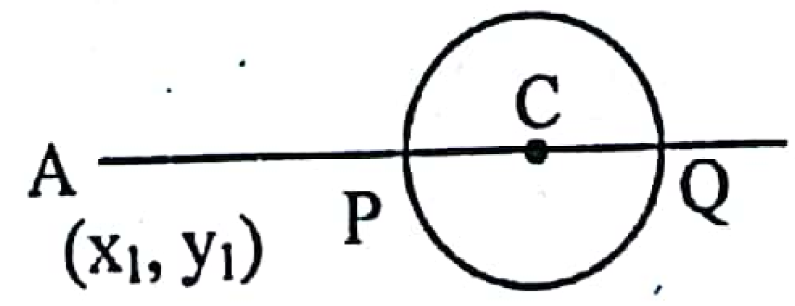
(a) Let the circle is $x^2+y^2+2gx+2fy+c=0$ and the point is (x_1, y_1) then

Point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as

$S_1 \Rightarrow x_1^2+y_1^2+2gx_1+2fy_1+c >, =, < 0$

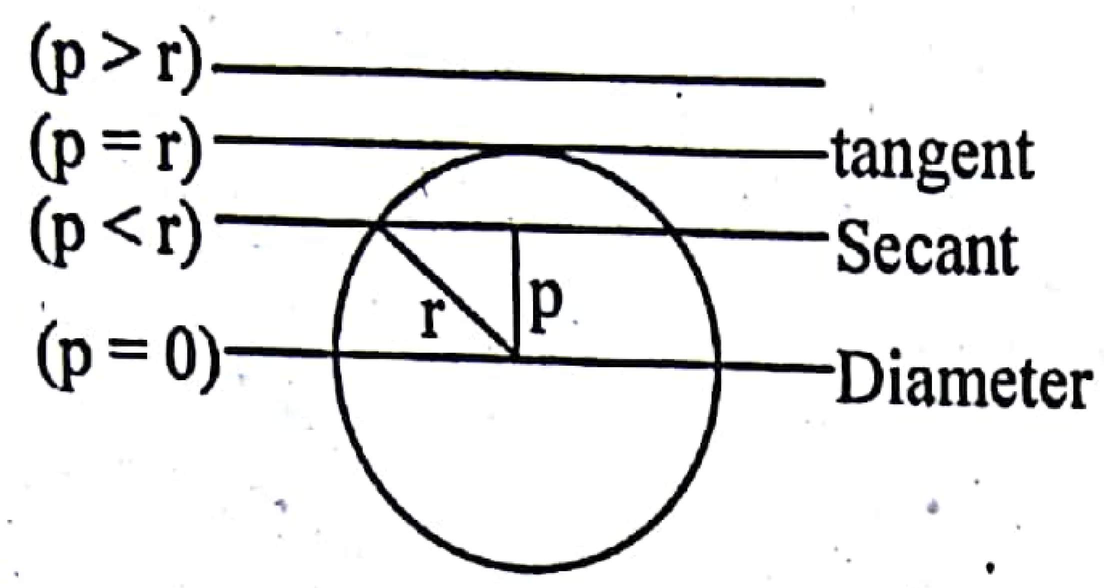
or $S_1 >, =, < 0$ (outside, on, inside)

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.



9. Line and Circle

Let L = 0 be a line and S = 0 be a circle. If 'r' be the radius of a circle and p be the length of perpendicular from the centre of circle on the line. Then if



$p > r \Rightarrow$ Line is outside the circle

$p = r \Rightarrow$ Line touches circle (tangent)

$p < r \Rightarrow$ Line is the chord of circle (secant)

$p = 0 \Rightarrow$ Line is diameter of circle

Note:

(i) Length of the intercept made by the circle on the line is

$$2\sqrt{r^2 - p^2}$$

(ii) The length of the intercept made by the line $y = mx + c$ with the circle $x^2 + y^2 = a^2$ is

$$2\sqrt{\frac{a^2(1+m^2) - c^2}{1+m^2}}$$

9.1 Condition for tangency :

A line L = 0 touches the circle S = 0, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle. i.e. p = r. This is the condition of tangency for the line L = 0.

Circle $x^2 + y^2 = a^2$ will touches the line

$y = mx + c$, if $c = \pm a\sqrt{1+m^2}$

Again

(i) If $a^2(1 + m^2) - c^2 > 0$ line meet the circle at real and distinct points.

(ii) If $c^2 = a^2(1 + m^2)$ line will touch the circle.

(iii) If $a^2(1 + m^2) - c^2 < 0$ line will meet circle at two imaginary points.

9.2 Intercepts made on coordinate axes by the circle :

The intercept made by the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ on

(i) x-axis $= |x_1 - x_2| = 2\sqrt{g^2 - c}$

(ii) y-axis $= |y_1 - y_2| = 2\sqrt{f^2 - c}$

Note:

Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts

(i) x-axis in two real, coincident or imaginary points according as $g^2 >, =, < c$

(ii) y-axis in two real, coincident or imaginary points according as $f^2 > = < c$

10. Equation of tangent

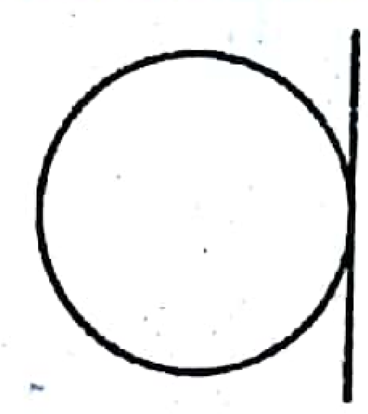
10.1 Equation of tangent at a point on a circle :

The equation of tangent to the circle $x^2 + y^2 = a^2$ at a point A (x_1, y_1) is $xx_1 + yy_1 = a^2$ or $T = 0$

Note :

If the equation of the circle are given in general form then the equation of tangent at A (x_1, y_1) of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$



10.2 Slope form :

The equation of the tangent to the circle $x^2 + y^2 = a^2$ in terms of its slope m is given by $y = mx \pm a\sqrt{1+m^2}$ and its point of contact is given by $\left(\mp\frac{am}{\sqrt{1+m^2}}, \pm\frac{a}{\sqrt{1+m^2}}\right)$

Note:

The equation of tangent to the circle $(x - h)^2 + (y - k)^2 = r^2$ in terms of its slope m is given by

$$y - k = m(x - h) \pm r\sqrt{1+m^2}$$

where $r = \sqrt{g^2 + f^2 - c}$

10.3 Tangent in parametric form:

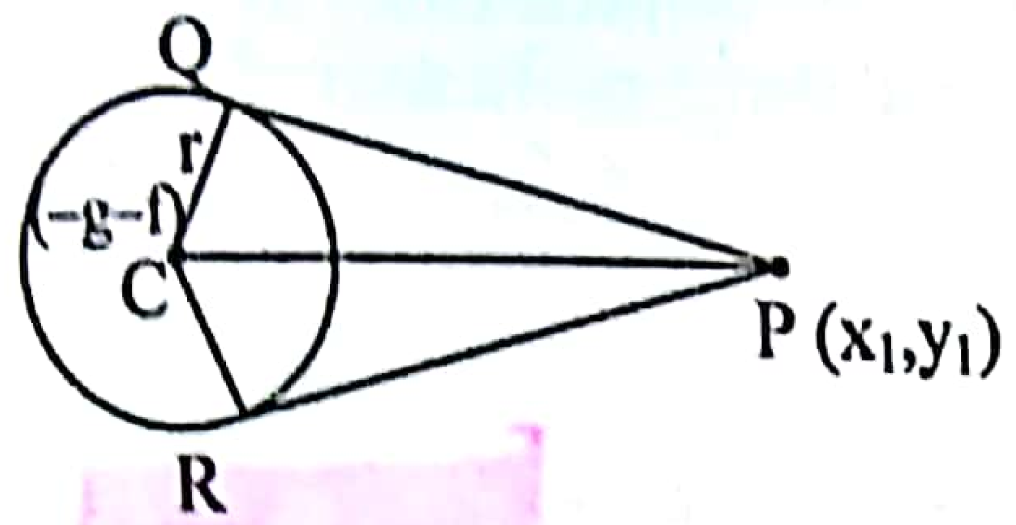
Let point is $P(a\cos\theta, a\sin\theta)$ then tangent is $x\cos\theta + y\sin\theta = a$.

11. Length of the tangent

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$S_1 = PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Length of tangent = $\sqrt{S_1}$



Note :

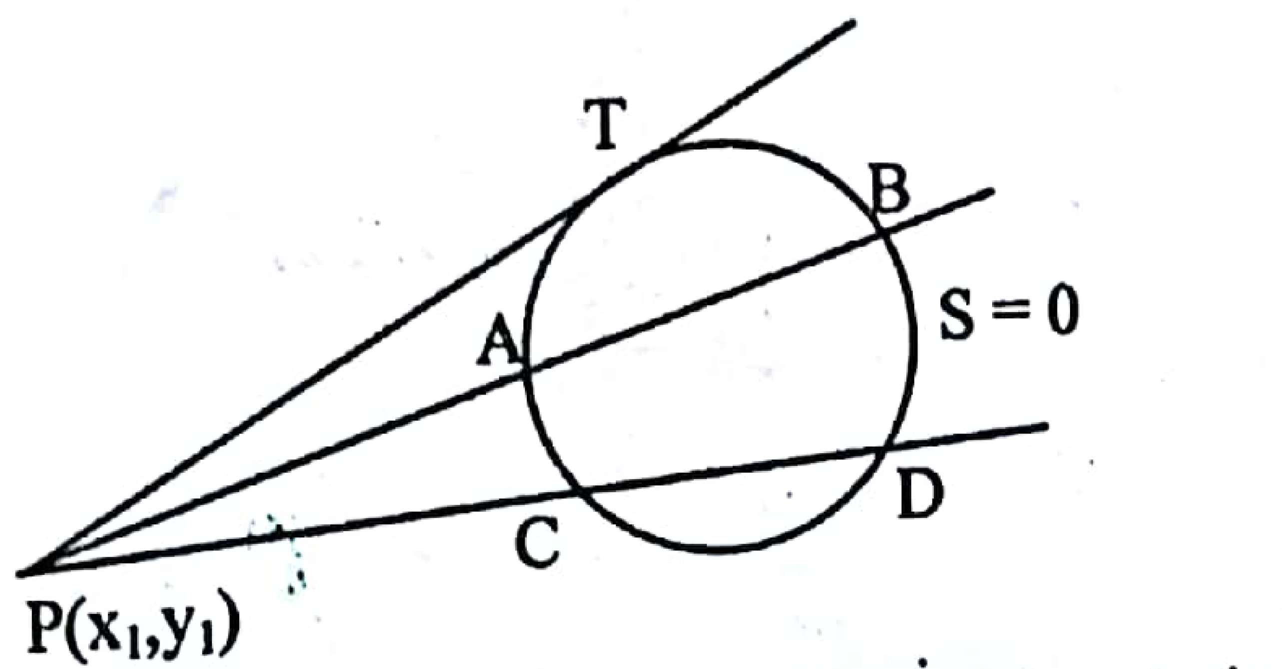
(i) If PQ is a length of the tangent from a point P to a given circle. Then PQ^2 is called the power of the point with respect to a given circle.

(ii) Area of quadrilateral PQCR = $r\sqrt{S_1}$ and angle between tangents PQ and PR is

$$\theta = 2\tan^{-1}\frac{r}{\sqrt{S_1}}$$

(iii) Power of a point w. r. t. a circle :

Let P (x_1, y_1) be point and secant (a line which cut the curve in two point) PAB drawn. The power of $P(x_1, y_1)$ w. r. t. $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to PA.PB which is $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$



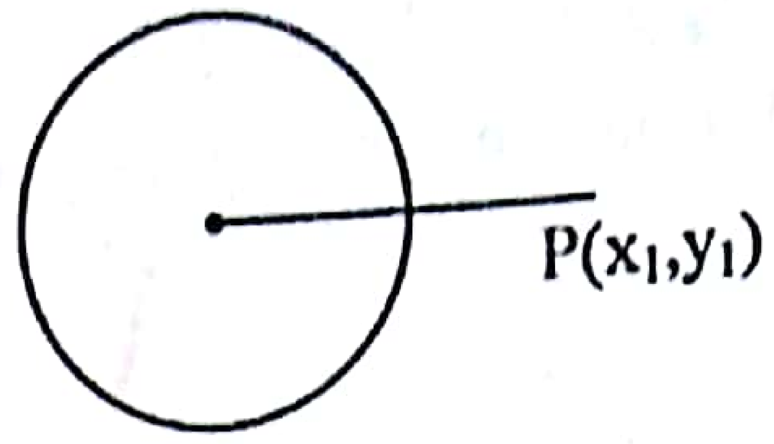
∴ Power remains constant for the circle i.e. independent of A and B

∴ PA . PB = PC. PD = PT^2 = square of the length of a tangent

[Power of a point P is positive, negative or zero according to position of P outside, inside or on the circle.

i.e. It gives the relative position of a point w. r. t. circle

i.e. $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >=< 0$]



12. Equation of Normal

The normal at any point on a curve is a line which is perpendicular to the tangent to the curve at that point.

Equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$(y - y_1) = \frac{y_1 + f}{x_1 + g}(x - x_1) \text{ or } \frac{y - y_1}{y_1 + f} = \frac{x - x_1}{x_1 + g}$$

Thus the condition of normality for the line $y = mx + d$ to be normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $mg = f + d$

Note :

(i) The equation of the normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is

$xy_1 - x_1y = 0$

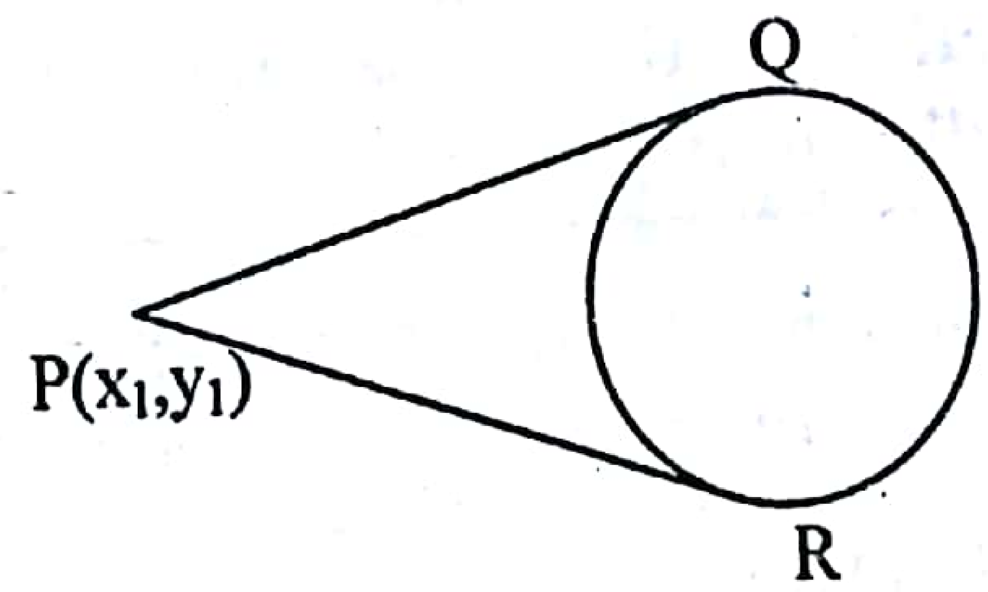
(ii) Normal to the circle always passes through the centre of the circle.

13. Pair of tangent

From a given point $P(x_1, y_1)$, two tangents PQ and PR can be drawn to the circle

$S = x^2 + y^2 + 2gx + 2fy + c = 0$.

Their combined equation is $SS_1 = T^2$ where S = 0 is the equation of the circle, T = 0 is the equation of the tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.



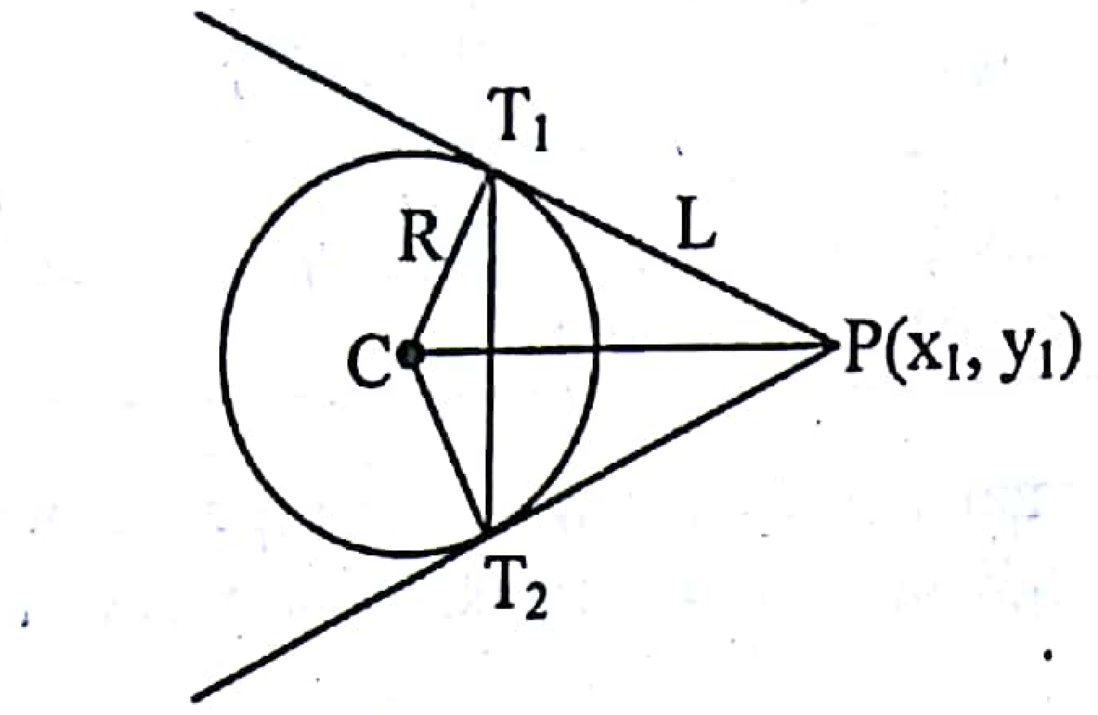
14. Chord of contact

A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(i.e. $T = 0$ same as equation of tangent)



Note :

(a) Length of chord of contact

$$T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$$

(b) Area of the triangle formed by the pair of the tangents & its chord of contact $= \frac{RL^3}{R^2 + L^2}$, where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(c) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$.

(d) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0.$$

(e) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $SS_1 = T^2$, where $S = x^2 + y^2 + 2gx + 2fy + c$;

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

15. Equation of the chord with a given middle point ($T = S_1$)

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Note :

The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

16. Director circle

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h, k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$

i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow [(h^2 + k^2 - a^2) - h^2] + [(h^2 + k^2 - a^2) - k^2] = 0$

$\Rightarrow h^2 + k^2 = 2a^2$

$\therefore$ locus of (h, k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

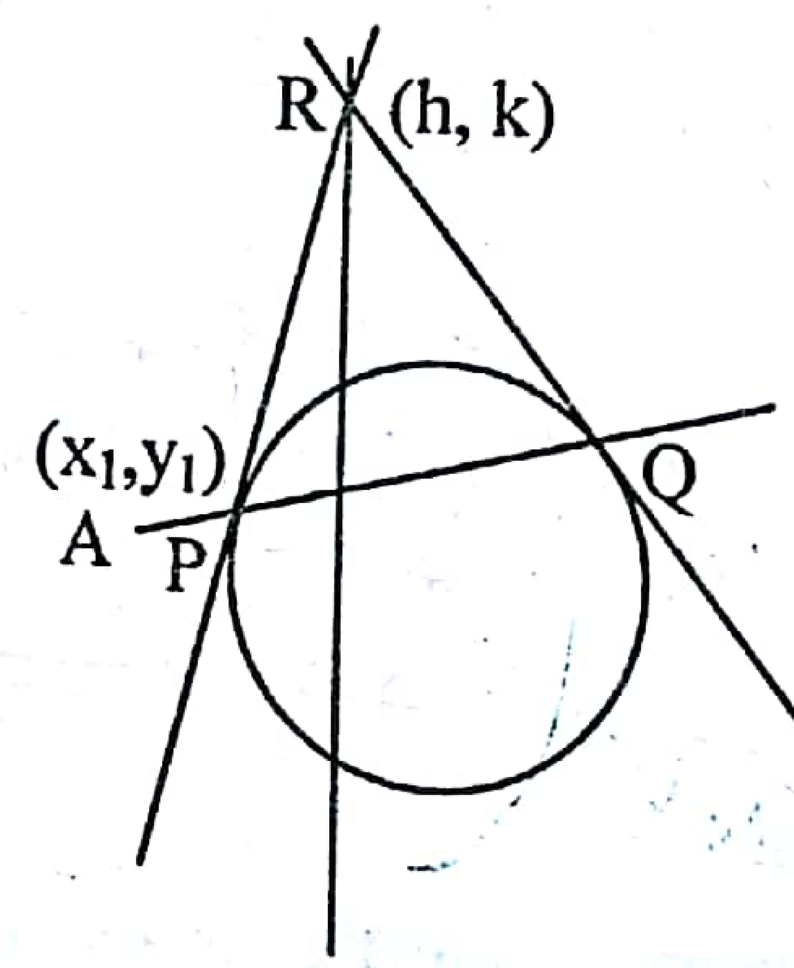
$\therefore$ Director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note :

The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

17. Pole & Polar

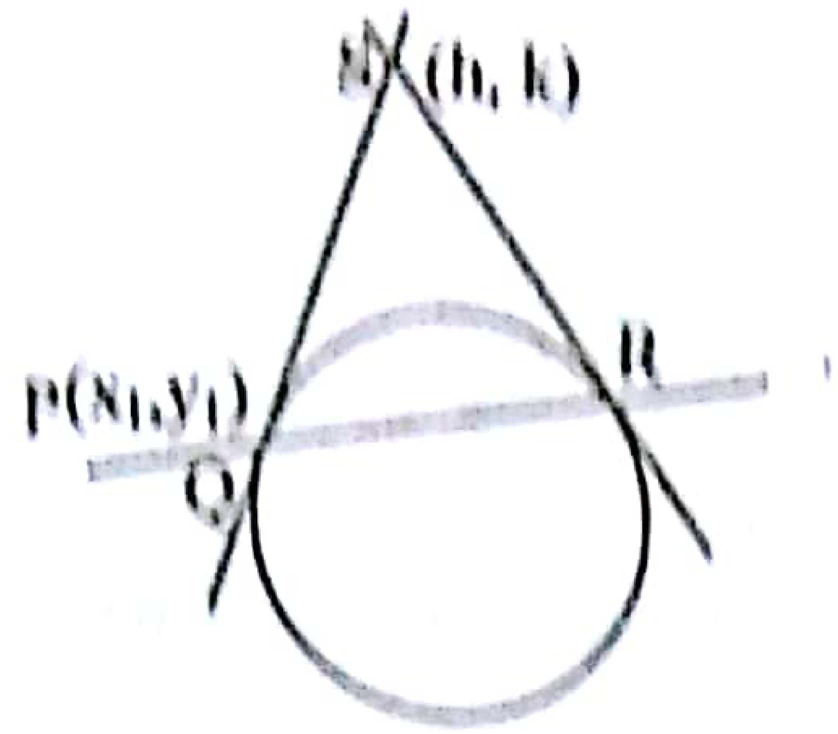
Let any straight line through the given point $A(x_1, y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



(a) The equation of the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 = a^2$:

Let PQR is a chord which passes through the point $P(x_1, y_1)$ which intersects the circle at points Q and R and the tangents are drawn at points Q and R meet at point S(h, k) then equation of QR the chord of contact is $x_1h + y_1k = a^2$

$\therefore$ locus of point S(h, k) is $xx_1 + yy_1 = a^2$ which is the equation of the polar.



Note :

(i) The equation of the polar is the T = 0, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar, chord of contact and tangent on that point are same.

(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P & Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(b) Pole of a given line with respect to a circle :

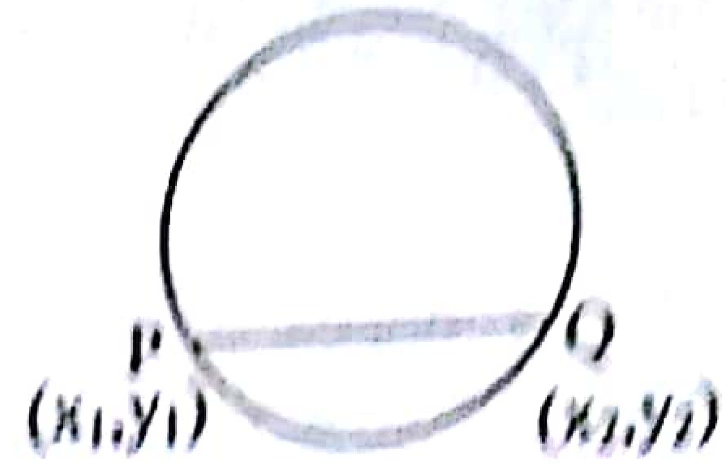
To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$ w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

18. Equation of a chord joining two point on the curve

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be a two point on a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Then the equation of chord joining any two point on the curve is given by

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0$$



Note :

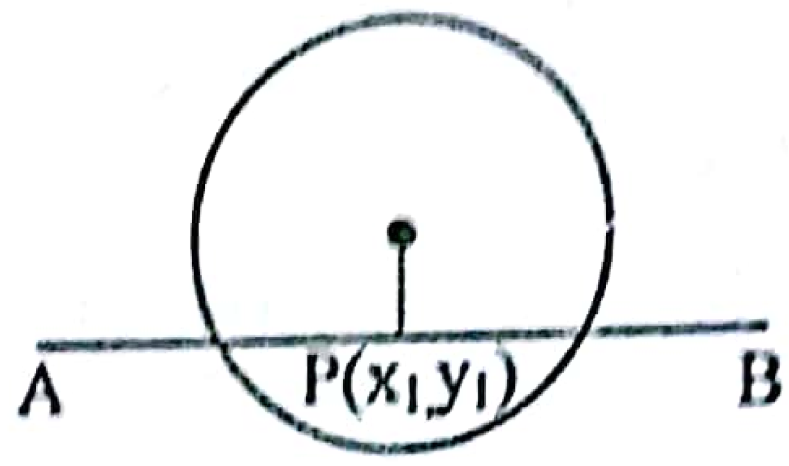
Parametric form : Chord joining two points whose parameters are ϕ_1 & ϕ_2 is

$$x\cos\left(\frac{\phi_1 + \phi_2}{2}\right) + y\sin\left(\frac{\phi_1 + \phi_2}{2}\right) = a\cos\left(\frac{\phi_1 - \phi_2}{2}\right)$$

18.1 Equation of chord of a circle in terms of its mid point :

The equation of chord of a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is

$$(y - y_1) = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$



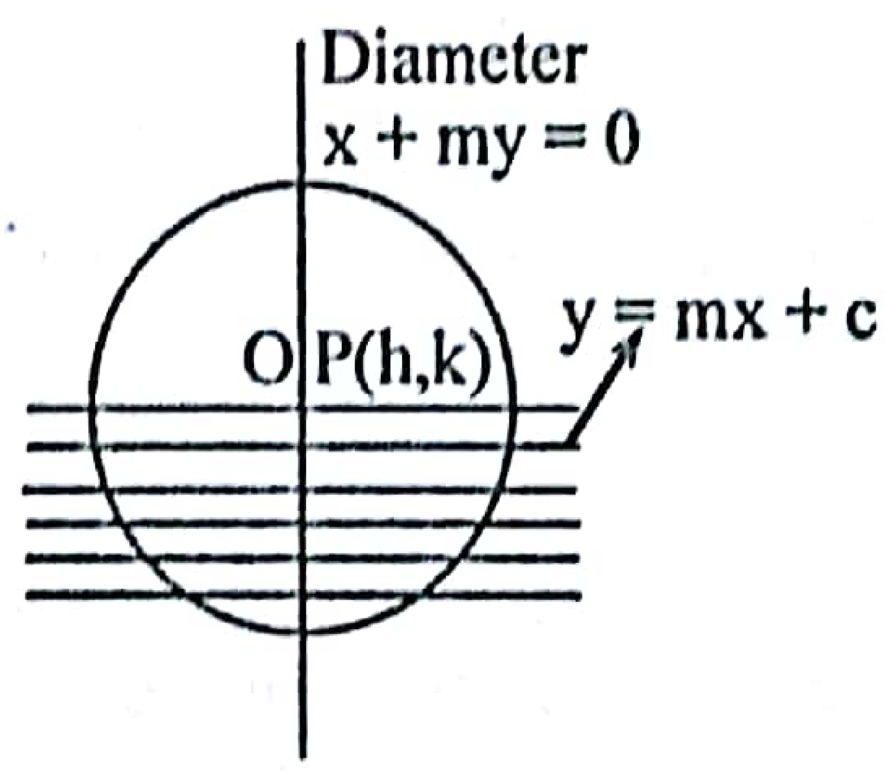
which on simplification give

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

which is designate as $T = S_1$

19. Diameter of a circle

The locus of middle points of a system of parallel chords of a circle is called the diameter of a circle.



The diameter of the circle $x^2 + y^2 = r^2$ corresponding to the system of parallel chords $y = mx + c$ is $x + my = 0$

Note :

(i) Every diameter passes through the centre of the circle.

(ii) A diameter is perpendicular to the system of parallel chords.

20. Family of circles

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is $S_1 + K S_2 = 0$ $(K \neq -1)$

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :$(x - x_1)(x - x_2) +$

$$(y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ where}$$

K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

21. System of two circles

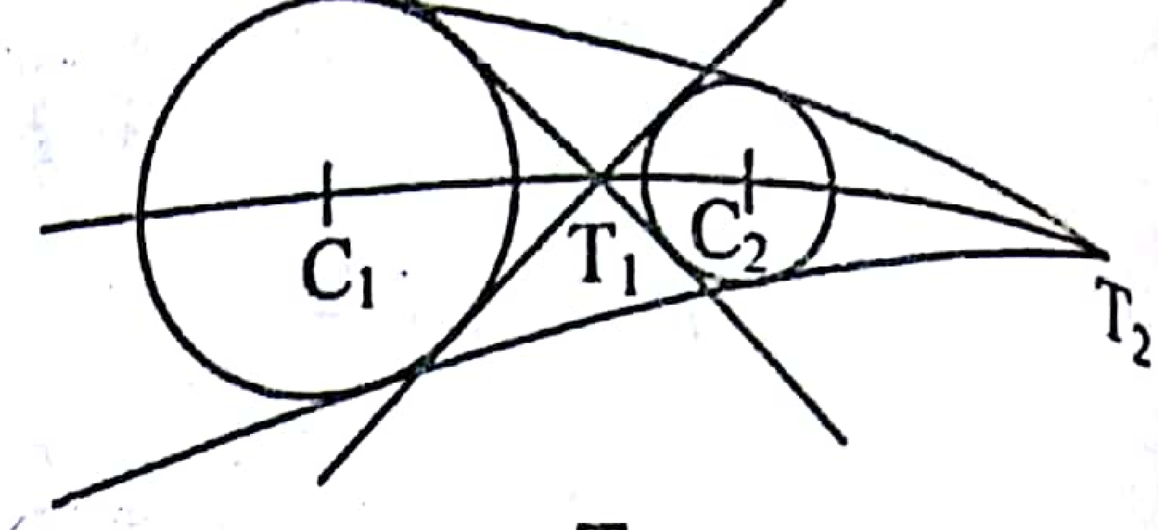
21.1 Relational position of two circles :

Let $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$ be the centre of two circle and r_1, r_2 be their radius then

Case I :

When $C_1C_2 > r_1 + r_2$ i.e. the distance between the centres is greater than the sum of their radii.

In this case , the two circles do not intersect with each other and four common tangents be drawn. Two common tangents intersects at T_2 called the **direct common tangents** and other two intersect at T_1 called the **transverse common tangents.**

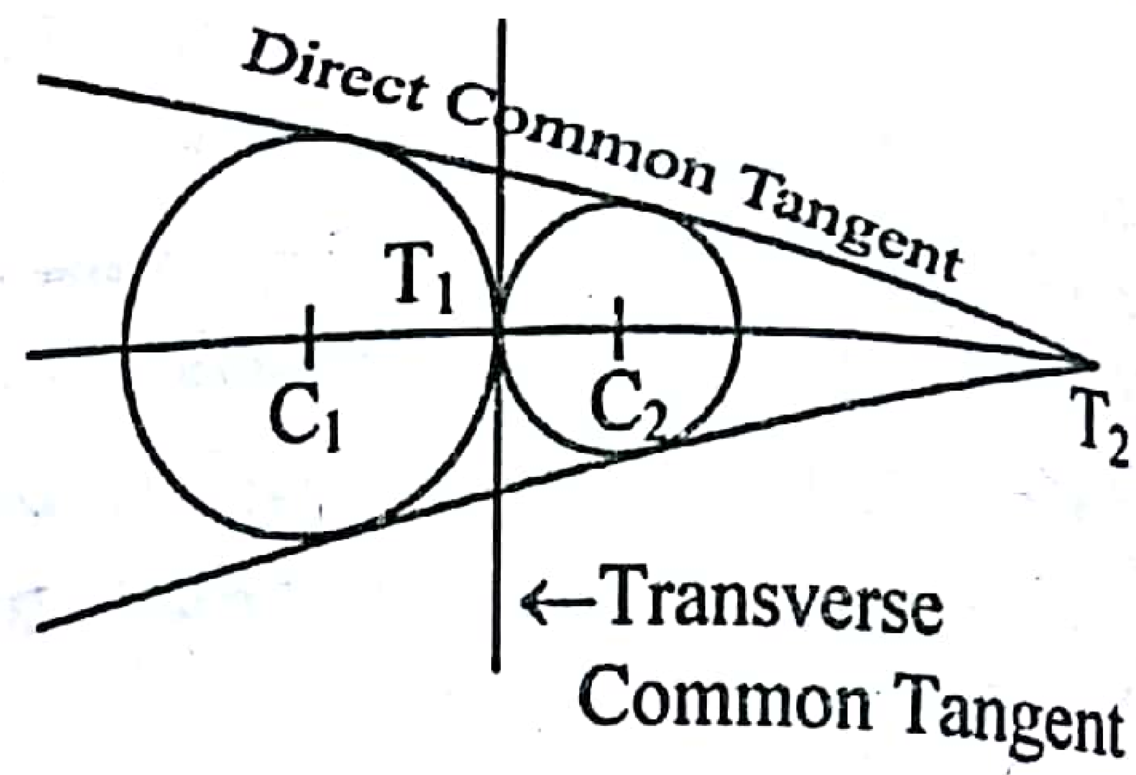


No. of Common Tangents : 4

Case II :

When $C_1C_2 = r_1 + r_2$ i.e. the distance between the centres is equal to the sum of their radii.

In this case, two direct tangents are real and distinct while the transverse tangents are coincident. The point T_1 divides C_1 and C_2 in the ratio of $r_1 : r_2$.

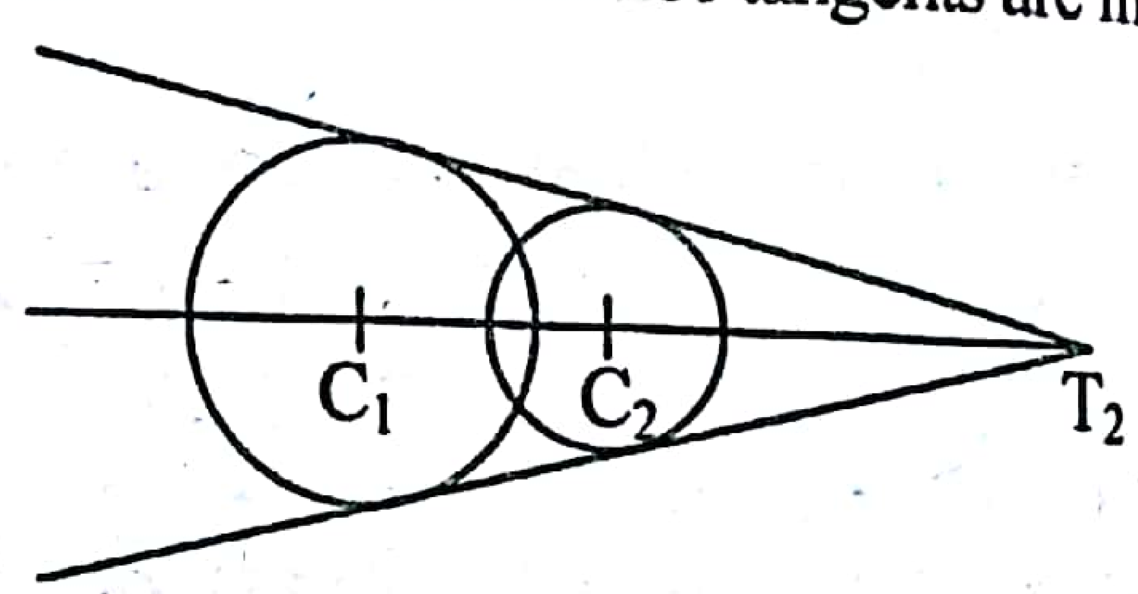


No. of Common Tangents : 3

Case III :

When $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ i.e. the distance between the centre is less than the sum of their radii.

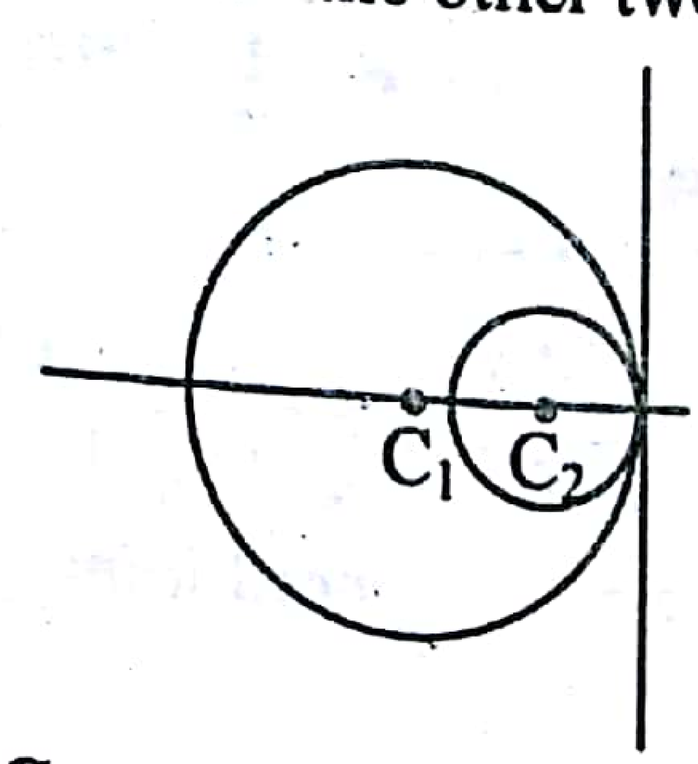
In this case, the two direct common tangents are real while the transverse tangents are imaginary.



No. of Common Tangents : 2

Case IV :

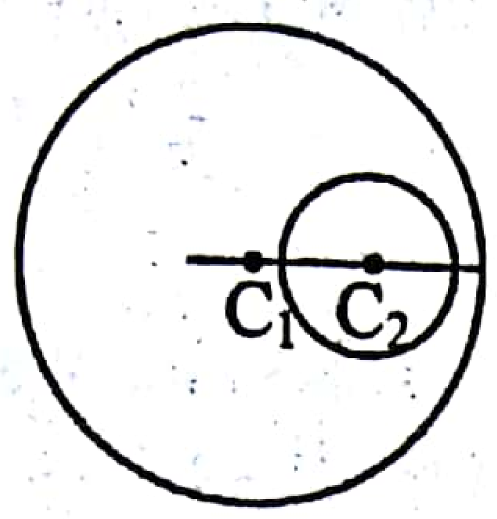
When $C_1C_2 = |r_1 - r_2|$ i.e. the distance between the centre is equal to the difference of their radii. In this case, two tangents are real and coincident while the other two are imaginary.



No. of Common Tangent : 1

Case V :

When $C_1C_2 < |r_1 - r_2|$ i.e. the distance between centre is less than the difference of their radii. In this case, all the four common tangents are imaginary.



No. of Common Tangent : 0

Note :

(i) Points of intersection of common tangents: The points T_1 and T_2 (points of intersection of indirect and direct common tangents) divides C_1C_2 internally and externally in the ratio $r_1 : r_2$.

(ii) Equation of the common tangents at point of contact : $S_1 - S_2 = 0$.

(iii) Point of contact : The point of contact divides C_1C_2 in the ratio $r_1 : r_2$ internally or externally as the case may be.

(iv) Length of direct common tangent

$$= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$$

(v) Length of transverse common tangent

$$= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$$

22. Common chord of two circle

The chord joining the points of intersection of two given circles is known as Common chord.

22.1 Equation of common chord :

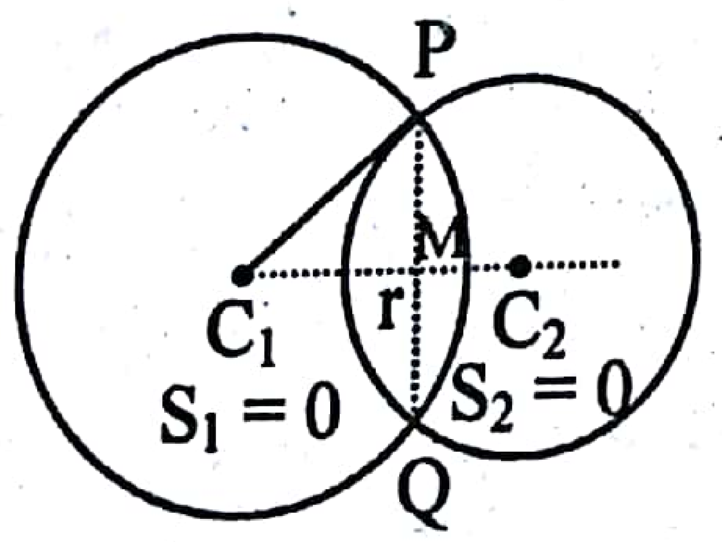
The equation of the common chord of two circles

$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$

or $S_1 - S_2 = 0$



and the length of the common chord

$PQ = 2\ PM = 2\ \sqrt{C_1P^2 - C_1M^2}$

Where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the ⊥ from centre C_1 to the common chord PQ.

23. The angle of intersection of two circles

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are

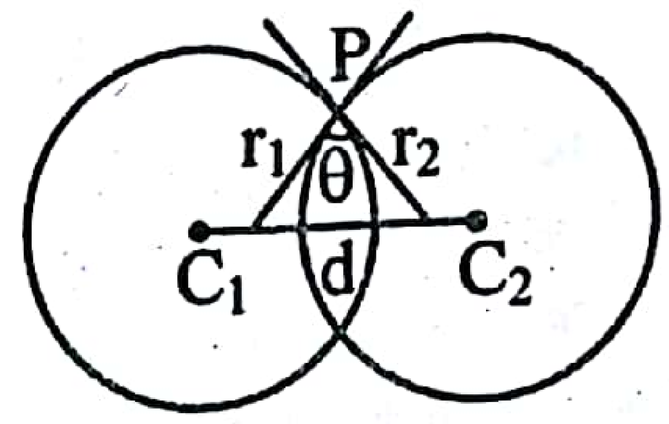
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the angle between them

then $\cos\theta = \dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}$

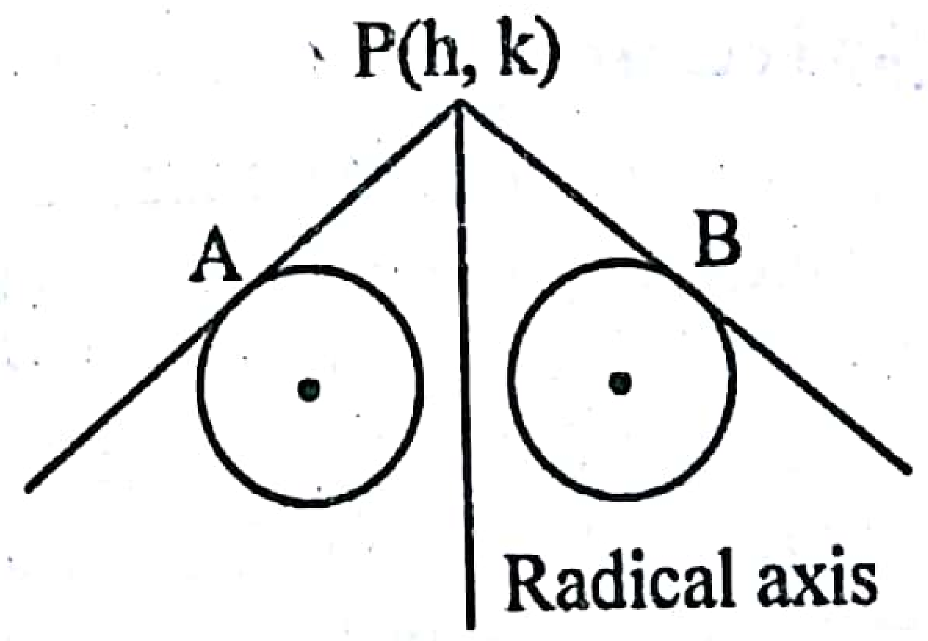
or $\cos\theta = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.



24. Radical axis of the two circles ($S_1 - S_2 = 0$)

(a) Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are



$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let P(h, k) is a point and PA, PB are length of two tangents on the circles from point P, then from definition

$\sqrt{h^2+k^2+2g_1h+2f_1k+c_1}$

$= \sqrt{h^2+k^2+2g_2h+2f_2k+c_2}$

or $2(g_1-g_2)h+2(f_1-f_2)k+c_1-c_2=0$

$\therefore$ locus of (h, k) is

$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0$

$S_1-S_2=0$

which is the equation of radical axis.

Note :

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$

(ii) If circles touch each other then Radical axis is the common tangent to both the circles.

(iii) When the two circles intersect on real point then common chord is the Radical axis of the two circles.

(iv) The Radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(v) The Radical axis of three circles (Taking two at a time) meet on a point.

(vi) If circles are concentric then the radical axis does not always exist but if circles are not concentric then Radical axis always exists.

(vii) If two circles are orthogonal to the third circle then Radical axis of both circle passes through the centre of the third circle.

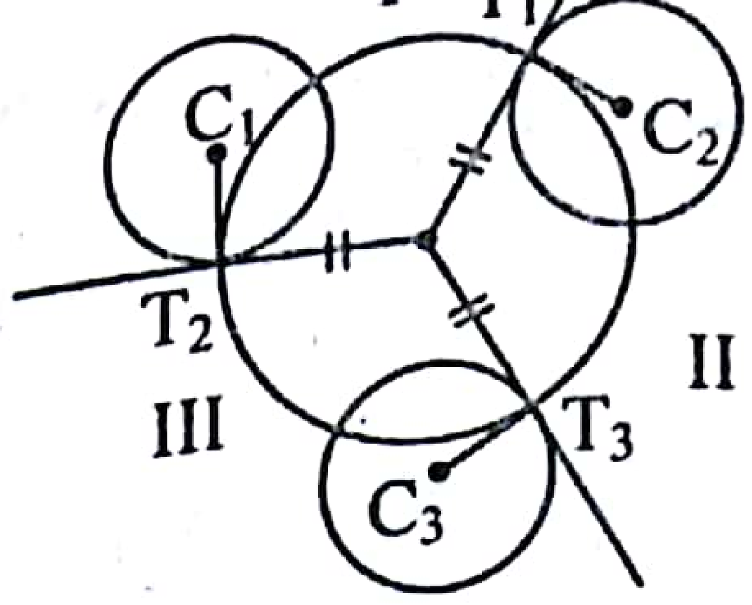
(viii) A system of circle, every pair of which have the same radical axis, is called a **coaxial system of circles.**

(b) Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles. To get the radical axis of three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ we have to solve any two $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$.

Note :

(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocentre will be its radical centre.

25. Orthogonality of two circles

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is $2g_1g_2+2f_1f_2=c_1+c_2$

Note :

(a) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

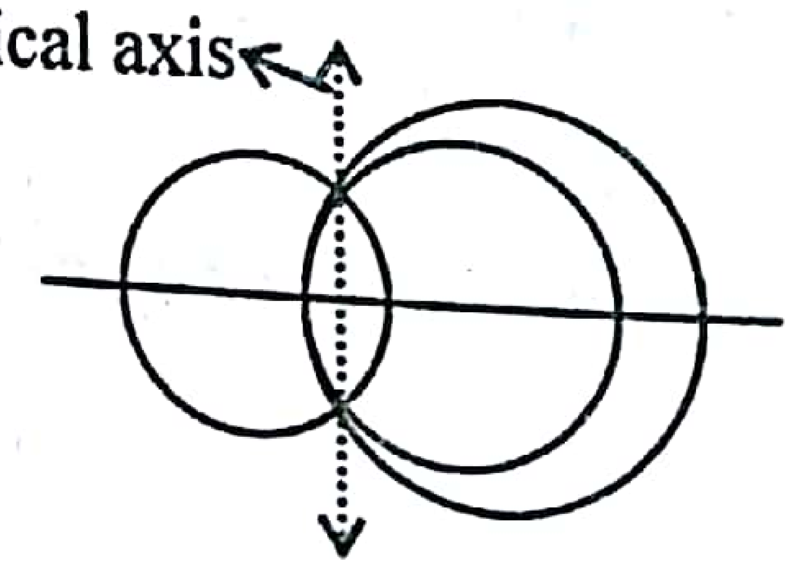
(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other and of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$, & $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

26. Coaxial system of circles

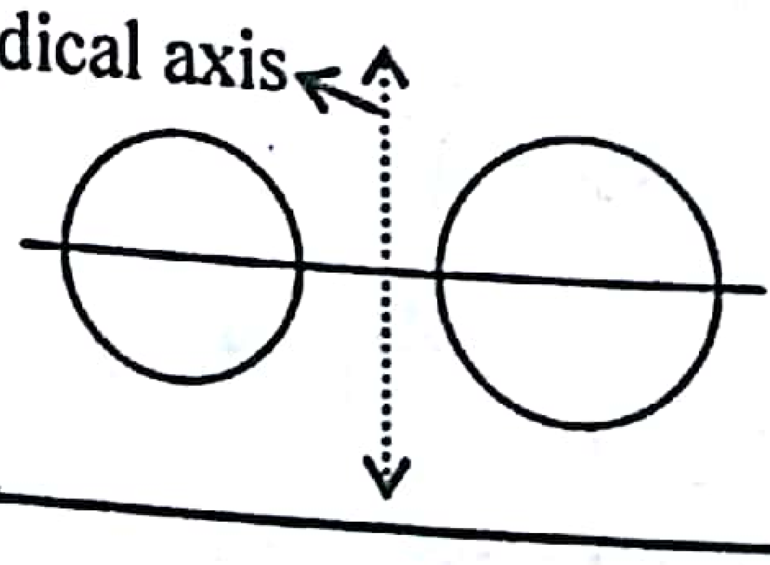
A system of circles is said to be coaxial when they have common radical axis, i.e. when radical axis of each pair of circles of the system is the same.

The radical axis of any pair of the circles is perpendicular to the line joining their centres, it follows that the centres of the circles of a coaxial system must be collinear.

Radical axis



Radical axis

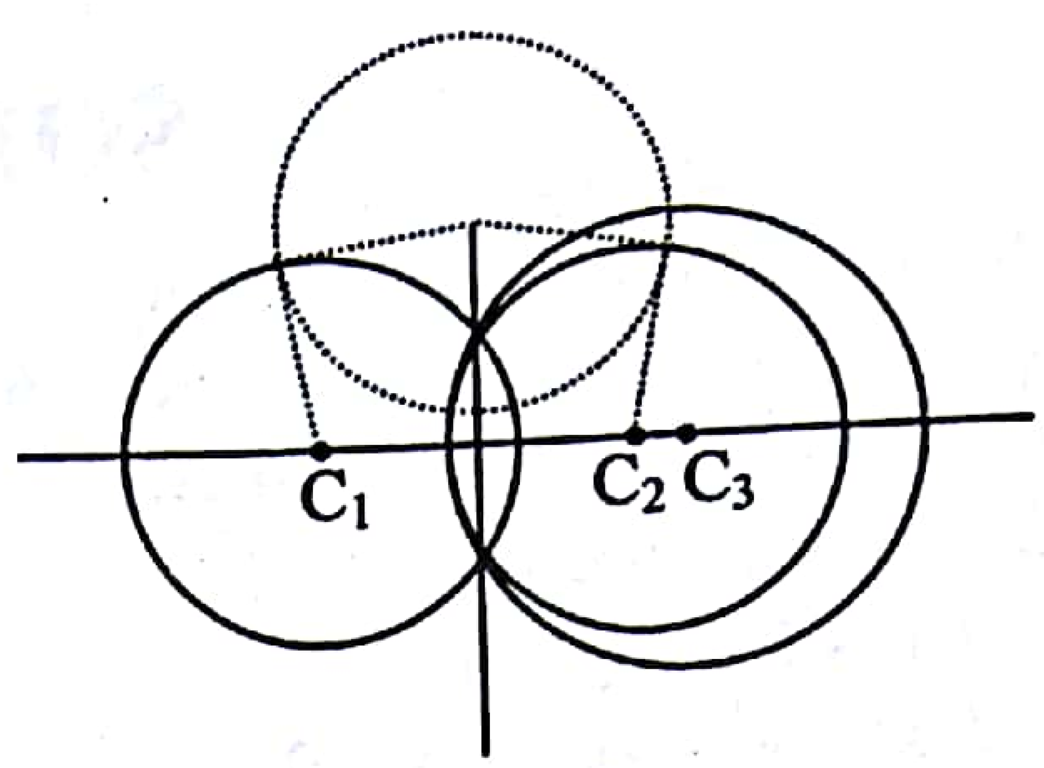


Note:

$S - S' = 0$ gives common radical axis of the system and in $S + \lambda u = 0$, family of circles, $u = 0$ is common radical axis of the system represents co-axial system.

Also $x^2 + y^2 + 2\lambda x + c = 0$ and

$x^2 + y^2 + 2\mu y + c = 0$ represent co-axial system (where λ and μ are parameters) having the y-axis and the x-axis as their common radical axis of the system respectively.



26.1 Limiting point of a coaxial system :

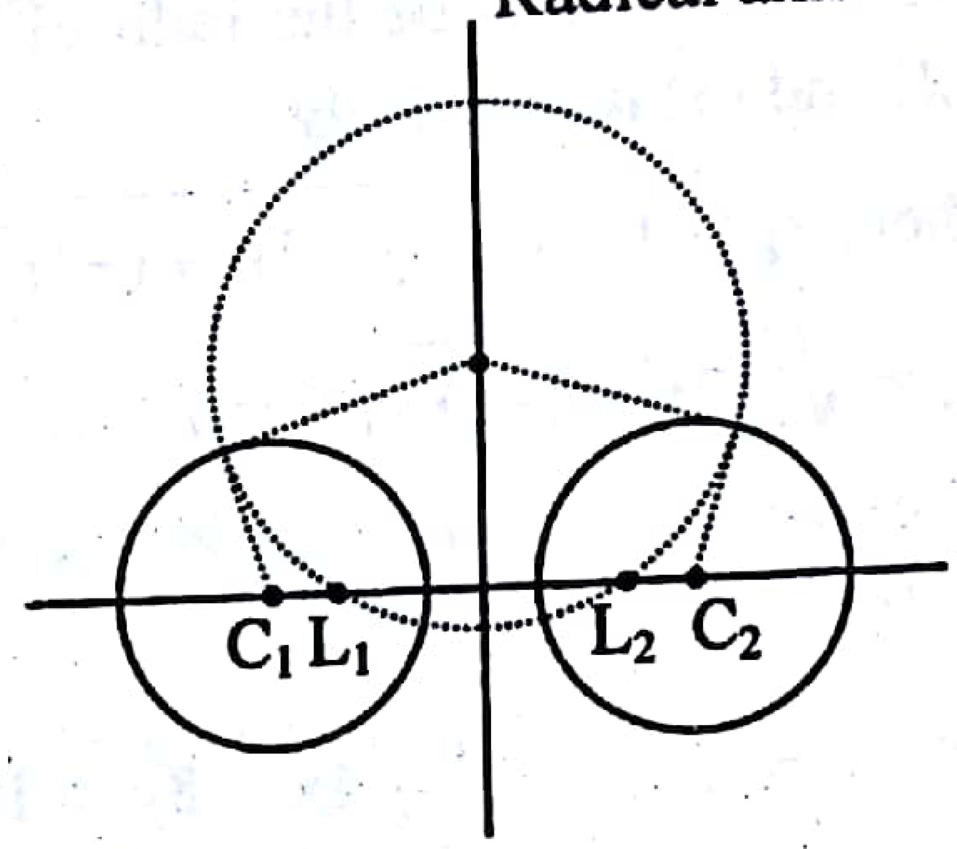
The equations $x^2 + y^2 + 2\lambda x + c = 0$ or $x^2 + y^2 + 2\mu y + c = 0$ represents a circle of system whose centre is $(-\lambda, 0)$ or $(0, -\mu)$ and whose radius is $\sqrt{\lambda^2 - c}$ or $\sqrt{\mu^2 - c}$.

This radius vanishes when $\lambda = \pm\sqrt{c}$ or $\mu = \pm\sqrt{c}$

Hence at the particulars points $(\pm\sqrt{c}, 0)$ or $(0, \pm\sqrt{c})$ we have point circles which belong to the system. These point circles are called the limiting points of the coaxial system.

26.2 Orthogonal circles of a coaxial system :

A set of coaxial circles can be cut orthogonally by another set of coaxial circles, the centres of each set lying on the radical axis of the other set, also one set is of the limiting point species and other set of the other species. The limiting points are therefore the intersection with the line of centres of any circle whose centre is on the common radical axis and whose radius is the tangent from it to any of the circles of the system.



Here limiting points are imaginary thus orthogonal circles do not meet the line of centres in real points. Here they pass through limiting points L_1 and L_2.

KEY CONCEPTS

1. Conic section

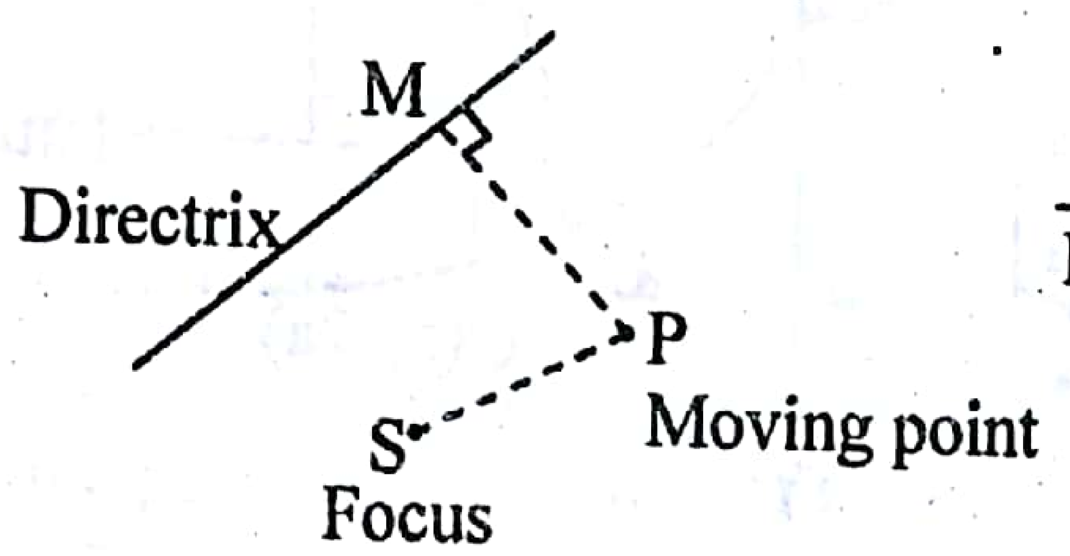
This chapter focuses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone not necessarily right circular can be cut in various ways by a plane, and thus different types of conic sections are obtained.

Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

Conic sections :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.



$$\frac{PS}{PM} = e$$

- The line passing through the focus & perpendicular to the directrix is called the **Axis.**
- A point of intersection of a conic with its axis is called a **Vertex.**

If S is (p, q) & directrix is $\ell x + my + n = 0$

then $PS = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$ &

$$PM = \frac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$$

$$\frac{PS}{PM} = e$$

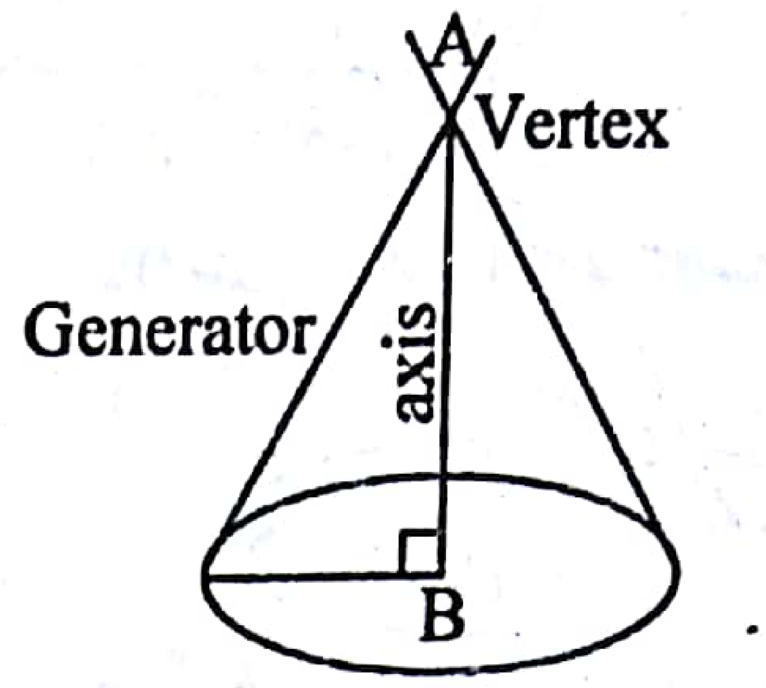
$\Rightarrow (\ell^2 + m^2)[(x-p)^2 + (y-q)^2] = e^2(\ell x + my + n)^2$

which is of the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

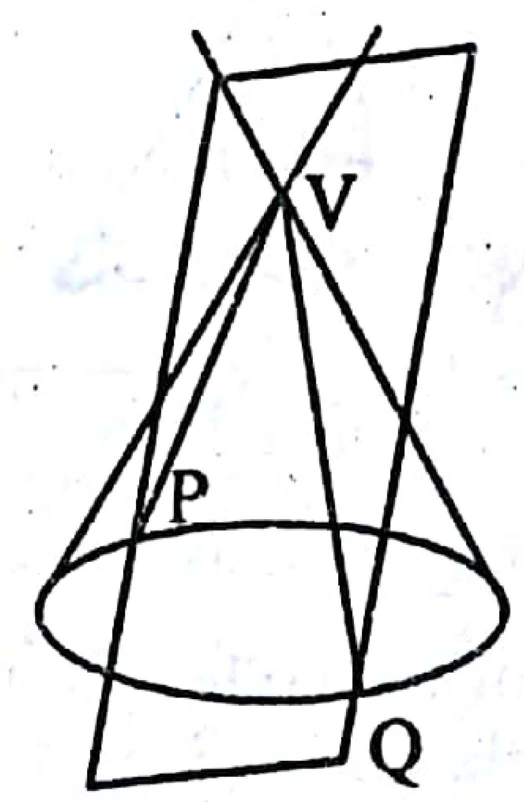
Section of right circular cone by different planes :

A right circular cone is as shown in the figure

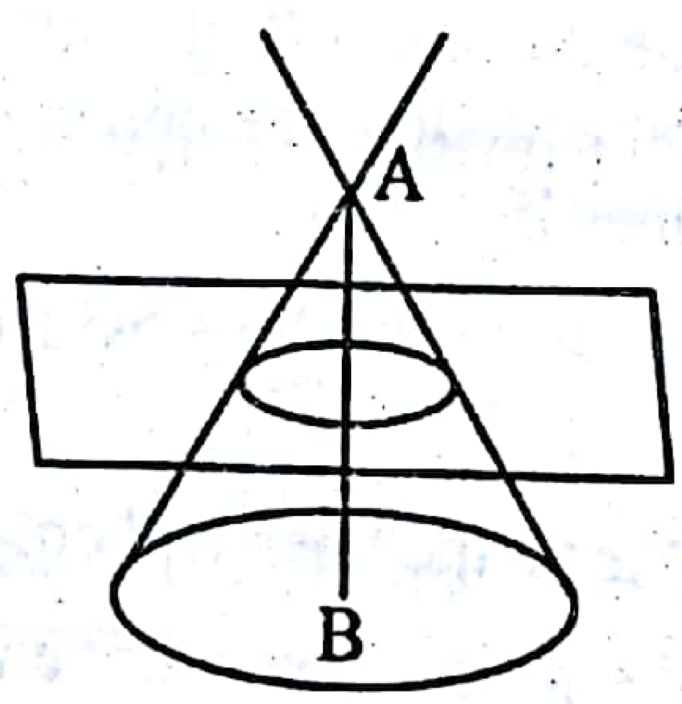


Circular Base

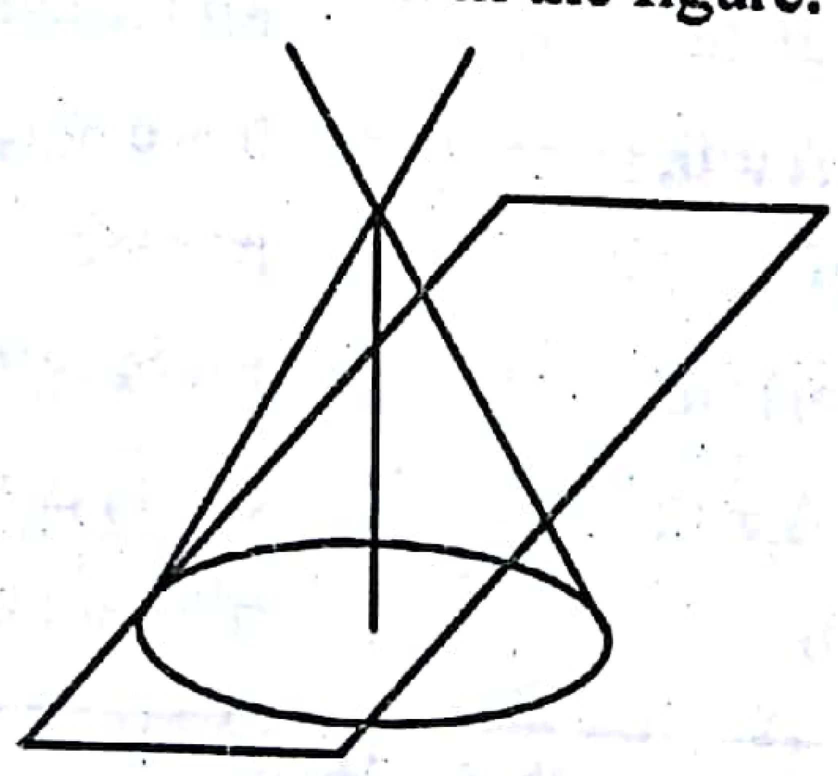
(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure.



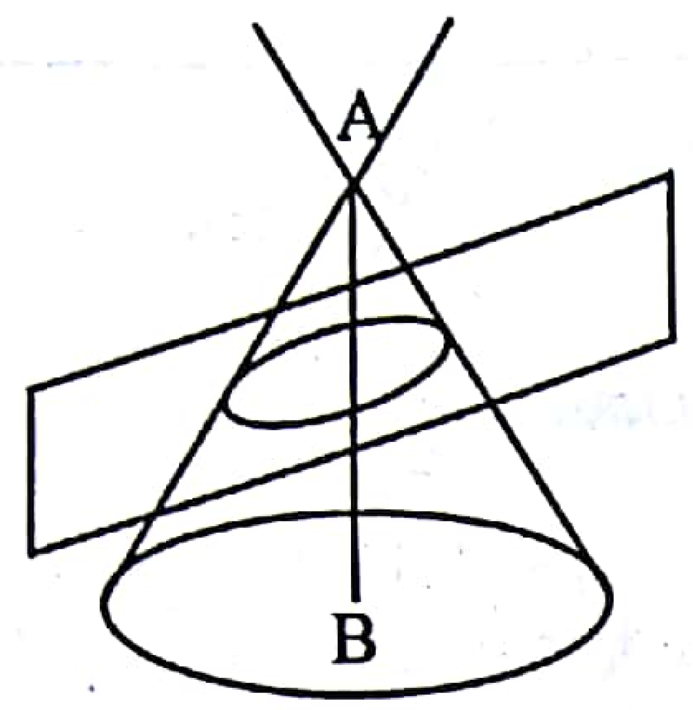
(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the figure.

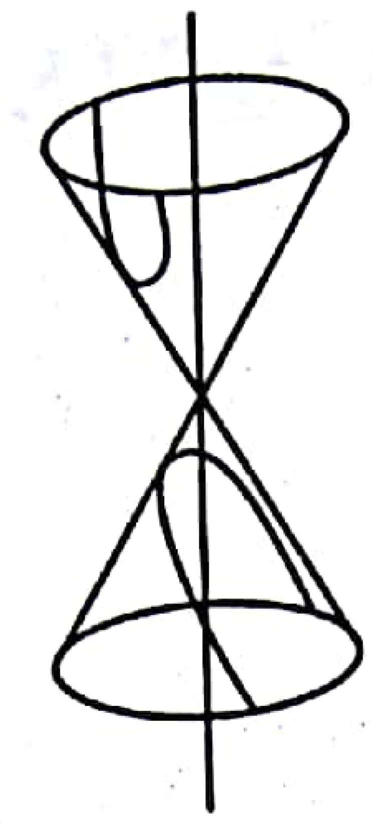


(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the figure.

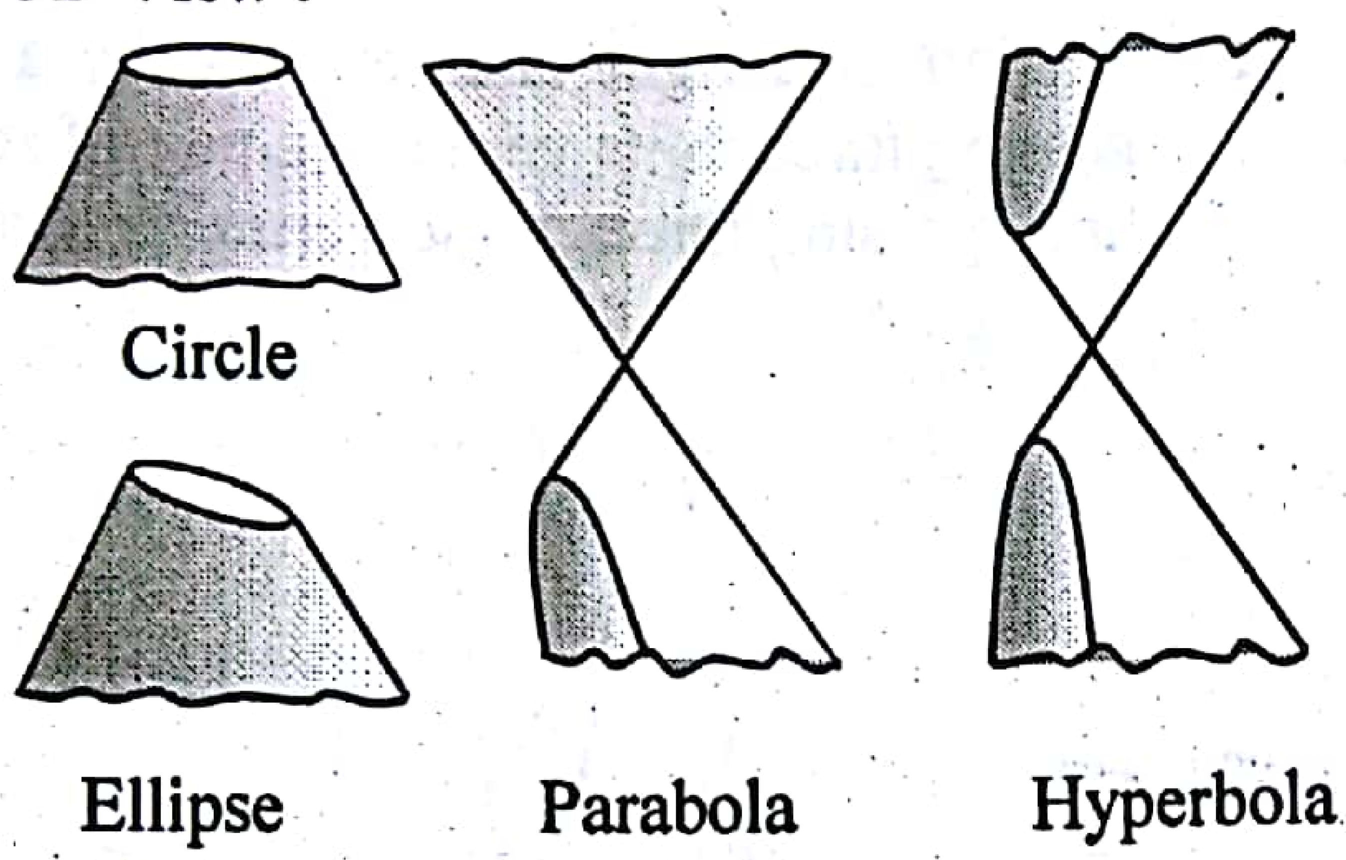


(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the figures.





3D View :



Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if :

$e > 1 \equiv h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1 \equiv h^2 \geq ab$ the lines will coincident.

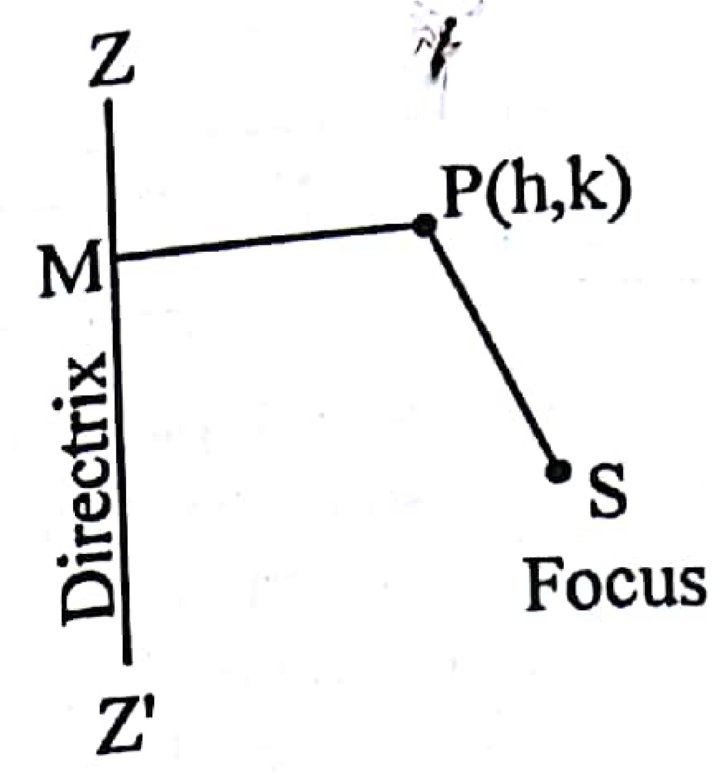
$e < 1 \equiv h^2 \leq ab$ the lines will be imaginary.

Case (ii) When the focus does not lie on directrix :

a parabola	**an ellipse**
$e = 1; \Delta \neq 0,$	$0 < e < 1; \Delta \neq 0;$
$h^2 = ab$	$h^2 < ab$
a hyperbola	**rectangular hyperbola**
$e > 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0$
$h^2 > ab$	$h^2 > ab; a + b = 0$

2. Parabola : Definition

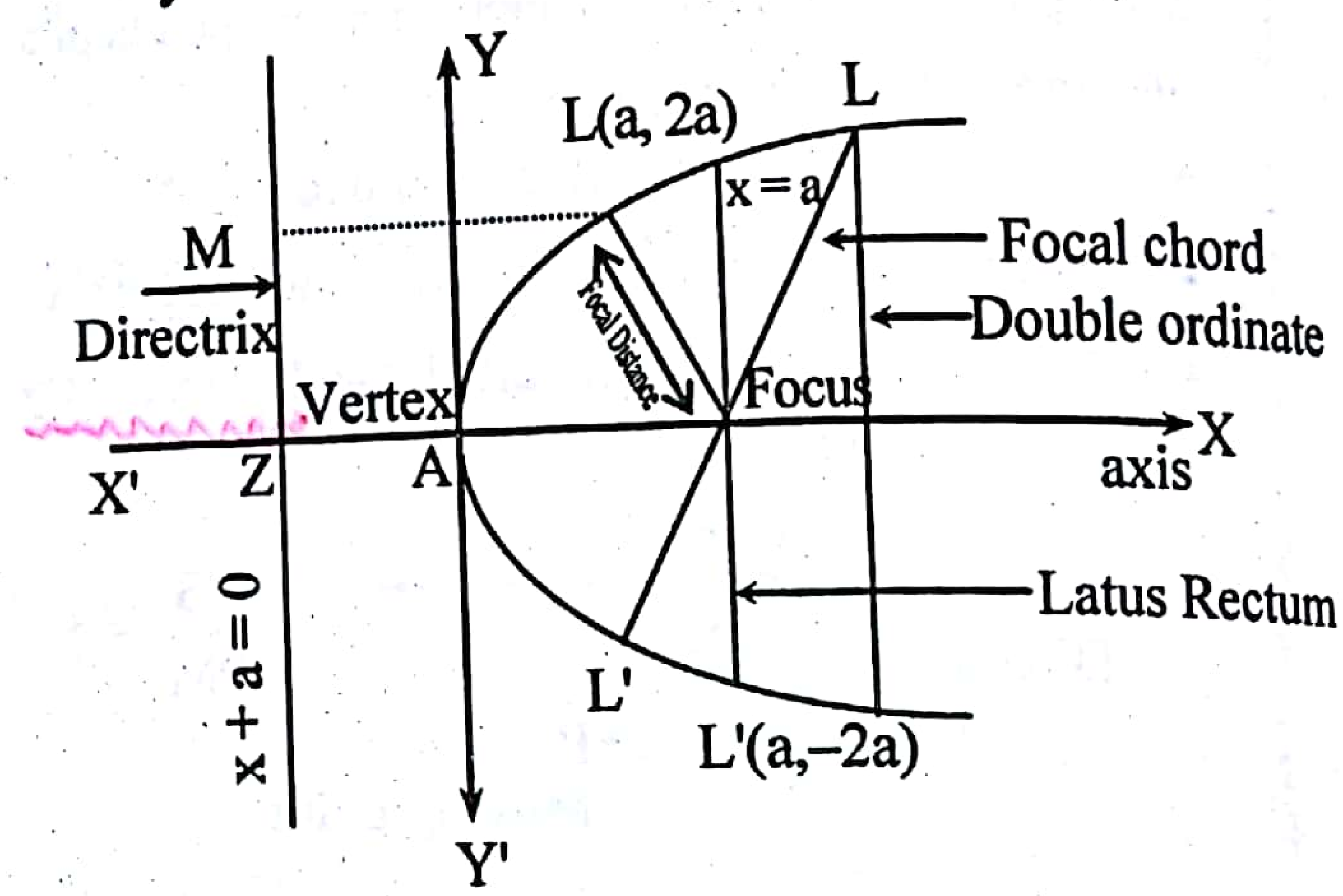
A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).



3. Equation of a parabola in its standard form

If we take vertex as the origin, axis as x-axis and distance between vertex and focus as 'a' then, the equation of the parabola in the simplest form will be :

$y^2 = 4ax.$



3.1 Parameters of the parabola $y^2 = 4ax$

(a) Vertex (A) $\Rightarrow (0, 0)$

(b) Focus (S) $\Rightarrow (a, 0)$

(c) Directrix $\Rightarrow x + a = 0$

(d) Axis $\Rightarrow y = 0$ or x-axis.

(e) Equation of Latus-Rectum $\Rightarrow x = a$

(f) Length of L.R. $\Rightarrow 4a$

(g) Ends of L.R. $\Rightarrow (a, 2a), (a, -2a)$

(h) The focal distance $\Rightarrow$ Sum of abscissa of the point and distance between vertex and L.R.= $|x_1 + a|$

(i) If length of any double ordinate of parabola $y^2 = 4ax$ is 2ℓ. Then co-ordinates of endpoints of this double ordinates are $\left(\frac{\ell^2}{4a}, \ell\right)$ and $\left(\frac{\ell^2}{4a}, -\ell\right)$.

6. Equation of the chord joining any two point on the parabola

Let $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be any two point on the parabola $y^2 = 4ax$. Then the equation of the chord joining these points is given by·

$y(t_1 + t_2) = 2x + 2at_1t_2$

Note:

(i) Length of the chord

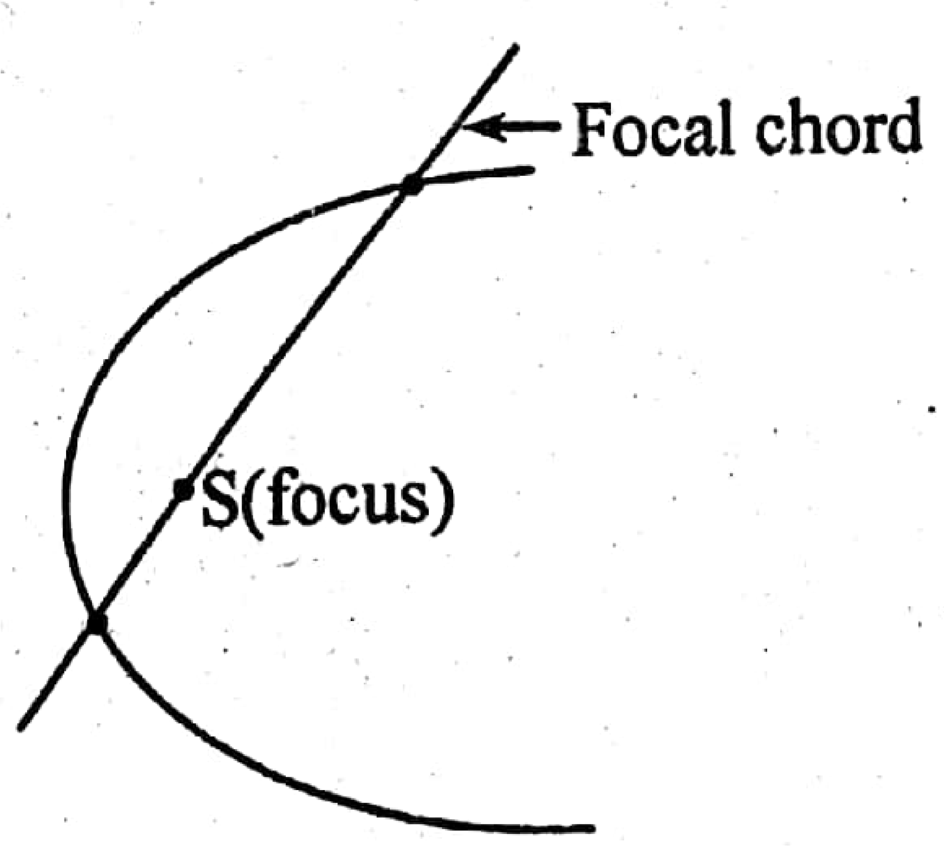
$= a(t_1 - t_2)\sqrt{(t_1 + t_2)^2 + 4}$

(ii) The length of the focal chord joining parameter t_1 and t_2 on the parabola $y^2 = 4ax$ is $a(t_2 - t_1)^2$

(iii) If ℓ_1 & ℓ_2 are the lengths of segments of a focal chord of a parabola. Then length of its latus rectum is $\dfrac{4\ell_1\ell_2}{\ell_1 + \ell_2}$.

(iv) Focal chord of minimum length is latus-rectum.

(v) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$, Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\dfrac{a}{t^2}, -\dfrac{2a}{t}\right)$.



(vi) Length of the focal chord making an angle α with the x-axis is $4a \operatorname{cosec} 2\alpha$.

7. Line & a Parabola

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $c <, =$ or $> a/m \Rightarrow$ condition of tangency is $c = a/m$

So the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = a/m$ & the point of contact is given by $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

Note:

If the line intercept the parabola. Then the length of the chord intercepted by the parabola on the line $y = mx + c$ is given by

$$\frac{4}{m^2}\sqrt{a(1 + m^2)(a - mc)}$$

8. Equation of tangent

8.1 Point form :

The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $T \equiv yy_1 = 2a(x + x_1)$

8.2 Parametric form :

The equation of tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $T \equiv ty = x + at^2$

Note:

The point of intersection of Tangents at t_1 and t_2 on the parabola is $[at_1t_2, a(t_1 + t_2)]$

8.3 Slope form :

The equation of a tangent of slope m to the parabola $y^2 = 4ax$ is given by $y = mx + a/m$ & the point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

9. Equation of Normal

9.1 Point form :

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by

$$(y - y_1) = \frac{-y_1}{2a}(x - x_1)$$

9.2 Parametric form :

The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is given by $y + tx = 2at + at^3$

9.3 Slope form :

The equation of normal to the parabola $y^2 = 4ax$ having slope m is $y = mx - 2am - am^3$ and its point of contact is given by $(am^2, -2am)$

Note:

(i) The equation of normal of slope m to $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$ are $y = mx + 2am + am^3$, $x = \dfrac{y}{m} - \dfrac{2a}{m} - \dfrac{a}{m^3}$ and $x = \dfrac{y}{m} + \dfrac{2a}{m} + \dfrac{a}{m^3}$ respectively.

$x^2 = 4ay$: → $y = mx + 2a + \dfrac{a}{m^2}$
$x^2 = -4ay$; → $y = mx - 2a - \dfrac{a}{m^2}$